MATH 282B Homotopy Theory

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This document contains course notes from MATH 282B (taught at Stanford, winter 2020) transcribed by Shintaro Fushida-Hardy. This was largely live-TeXed during lectures, but some additional thoughts and mistakes may have been added.

Contents

Chapter 1

Higher homotopy groups

1.1 Lecture 1

1.1.1 Admin

Office hours: Always happy to talk.

Books: Hatcher, Ralph Cohen's draft.

Most probable course outline:

- First half: Hatcher, chapter 4. Higher homotopy groups (of spheres).
    1. Fibre bundles.
    2. Cohomology. (In particular, stable homotopy.) Spectra and Brown representability.
- Spectral sequences
- Model categories and homotopy limits.
- Approximately two weeks of leeway, students can choose the direction.

1.1.2 Homotopy groups

Setting:

- All spaces are pointed (for now).
- X, Y denote spaces.

- $[X, Y]$ denotes homotopy classes of maps $X \to Y$. Then $[X \wedge Y, Z] = [X, Z^Y]$, and $[\Sigma X, Y] = [X, \Omega Y]$. More explicitly, the former is an adjunction between the "smash product with Y" and "maps from Y". The latter is an immediate consequence; an adjunction between the reduced suspension and loop space (since $\Sigma X := X \wedge \mathbb{S}^1$).

Definition 1.1.1. The higher homotopy groups of a space X are $\pi_n(X) := [\mathbb{S}^n, X]$.

For example, $\pi_n(\mathbb{R}^k) = 0$, $\pi_0(X) = \{\text{path components of } X\}$. $\pi_1(X)$ is the fundamental group.

Definition 1.1.2. Alternative definition:

$$\pi_n(X) = \{(D^n, \mathbb{S}^{n-1}) \to (X, *)\}/\text{homotopy}.$$

That is, maps sending $\mathbb{S}^{n-1}$ to $*$, where homotopies are invariant on $\mathbb{S}^{n-1}$.

Let $f, g \in \pi_n(X)$, $n \geq 1$. Then the above definition gives a clear group operation.

Proposition 1.1.3.
- For $n \geq 2$, homotopy groups are abelian. This can be seen via a diagrammatic proof. Explicitly, to prove that $f + g = g + f$ for arbitrary f and g, consider $f + g$. 1. Shrink the domains of f and g so that they are disjoint subdisks of D^n with room to be slid past each other. 2. Exchange their order by sliding them past each other. 3. Expand their domains to obtain $g + f$.
- If x_0, x_1 are connected by a path, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$.
- In fact, there is a canonical action $\pi_1(X) \curvearrowright \pi_n(X)$ given by the isomorphism above. X is *abelian* if the action is trivial.
- Each π_n is a functor from topological spaces to groups.

Proposition 1.1.4. Let $\widetilde{X} \to X$ be a covering space. Then $\pi_n(\widetilde{X}) \cong \pi_n(X)$ whenever $n \geq 2$.

Proof. Consider the induced map $\pi_n(\widetilde{X}) \to \pi_n(X)$ (induced by the covering map). Must prove it is surjective and injective. Surjectivity is immediate, since for each $n \geq 2$, $\mathbb{S}^n$ has trivial fundamental group. Thus the covering map lifts any map $\mathbb{S}^n \to X$. Similarly for injectivity, lift maps $I \times \mathbb{S}^n \to X$. $\square$

Remark. Covering spaces can be thought of in the following way. "A space that differs from the base space only in the first homotopy group, and no others."

Corollary 1.1.5. *Since $\mathbb{R} \to \mathbb{S}^1$ is a covering map, the homotopy groups of a circle are*

$$\pi_n(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 1.1.6.
$$\pi_n(\Pi_\alpha X_\alpha) = \Pi_\alpha(\pi_n(X_\alpha)).$$

Proof. This follows from the fact that maps $\mathbb{S}^n \to \Pi_\alpha X_\alpha$ are products of maps from $\mathbb{S}^n$ to X_α. $\square$

1.1.3 Relative homotopy groups and homotopy long exact sequence

Let $A \subset X$ be a subspace.

Definition 1.1.7. The relative homotopy groups are

$$\pi_n(X, A) = \{(D^n, \mathbb{S}^{n-1}) \to (X, A)\}/\text{homotopy}.$$

Here the homotopy (that is being modded out) is *not relative to* $\mathbb{S}^{n-1}$, but the homotopy restricted to $\mathbb{S}^{n-1}$ must have image contained in A.

Lemma 1.1.8. If $[f] \in \pi_n(X, A)$ and $[f] = 0$, then f is homotopic to g rel $\mathbb{S}^{n-1}$ with $\operatorname{im} g \subset A$.

Proof. Note that $0 \in \pi_n(X, A)$. $\Leftarrow$ is the easy direction: Suppose $f \sim g$, where $\operatorname{im} g \subset A$. Then $g : (D^n, \mathbb{S}^{n-1}) \to (A, A)$.

For the converse direction, suppose $[f] = 0$, so that it is homotopic to the constant map. Then $\operatorname{im} f \subset A$. Consider "thickening up" the boundary of f and applying the homotopy $f \sim 0$. $\square$

Remark. It is not true in general that $\pi_n(X, A) \cong \pi_n(X/A)$!

Theorem 1.1.9.

$$\cdots \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots$$

is a long exact sequence, where the first map is induced by the inclusion map, the second by $(D^n, \mathbb{S}^{n-1}) \to (X, *)$, and the third by restricting each element $(D^n, \mathbb{S}^{n-1}) \to (X, A)$ to the boundary, $\mathbb{S}^{n-1}$.

Proof. Exactness at $\pi_n(X)$: The kernel is the collection of maps $(D^n, \mathbb{S}^{n-1}) \to (X, *)$ which are homotopic to $(D^n, \mathbb{S}^{n-1}) \to (A, *)$ rel $\mathbb{S}^{n-1}$. By the previous lemma, this is just the image of the inclusion map. Exactness at $\pi_n(X, A)$ and $\pi_{n-1}(A)$ can also be verified directly. $\square$

1.2 Lecture 2

Example of homotopy long exact sequence: Let X be a space, and CX the cone of X:

$$CX \coloneqq X \times I / X \times \{0\}.$$

Then $CX \cong *$, so by the homotopy long exact sequence, $\pi_n(CX, X) \cong \pi_{n-1}(X)$.

Remark. $CX/X = \Sigma X$, but

$$\pi_n(\Sigma X) \ncong \pi_{n-1}(X).$$

If this were true, it would be easy to determine all the homotopy groups of spheres! On the other hand,

$$\pi_n(X) \cong \pi_{n-1}(\Omega X).$$

I.e. loops allow us to shift homotopy, but suspension does not.

Definition 1.2.1. X is *n-connected* if $\pi_k(X) = 0$, $k \neq n$. (X, A) is *n-connected* if $\pi_k(X, A) = 0$ for $k \leq n$.

1.2.1 Homotopy groups and CW-complexes, Whitehead's theorem

Recall: CW-complexes are "built from spheres". These should have good relations with homotopy groups.

Lemma 1.2.2. If $(X, A) \to (Y, B)$ is a map of CW-pairs, and $\pi_n(Y, B) = 0$ for all n, where $X \setminus A$ has cells, then $f \sim g$ rel A with $g : (X, A) \to (B, B)$.

Proof. Induction on cells of A. Idea: Cell of $X \setminus A$ is $(D^n, \mathbb{S}^{n-1}) \to (Y, B)$ by restriction. This is trivial in $\pi_n(Y, B)$, so it can be retracted to B. $\square$

Theorem 1.2.3 (Whitehead)**.** *Let $f : X \to Y$ be a map of CW-complexes, with $f_* : \pi_n(X) \to \pi_n(Y)$ an isomorphism for every n. Then f is a homotopy equivalence.*

CW-complexes are completely determined by homotopy groups! Gives some insight into why they are difficult to compute in general, since they are evidently so strong.

Proof. First assume $X \hookrightarrow Y$ is a CW-inclusion. By the homotopy long exact sequence, $\pi_n(Y, X)$ is zero. Thus apply above lemma to $(Y, X) \to (Y, X)$. For the general case, let $f : X \to Y$. This factors through the mapping cylinder: $X \to M_f \to Y$. The latter is a homotopy equivalence. We require the former inclusion to be cellular and apply the previous special case. By the succeeding theorem, we are done. $\square$

Example. Consider $\mathbb{S}^2 \times \mathbb{RP}^\infty$ and $\mathbb{RP}^2$. The former has a double cover; $\mathbb{S}^2 \times \mathbb{S}^\infty$, while the latter has a double cover; $\mathbb{S}^2$. Since $\mathbb{S}^\infty$ is contractible, both spaces have the same homotopy classes. However, they have different cohomology, so there is no map realising a homotopy equivalence. This is because we do not have isomorphisms between the homotopy groups induced from a map between the spaces, as required in the Whitehead theorem.

1.2.2 Cellular maps

Definition 1.2.4. A map $f : X \to Y$ between CW complexes is called *cellular* if $f(X^n) \subset Y^n$.

Theorem 1.2.5. *Every map $f : X \to Y$ is homotopic to a cellular map.*

Proof. Follows from an induction. The idea of the crucial step: suppose e^n is a cell of X with $f(e^n) \cap e^m \neq \varnothing$ where e^m is a cell of dimension $m > n$ of Y. We proceed by differential topology: f is homotopic to a smooth map. Restrict this smooth map to $e^n \to e^m \cup$ stuff. By counting (diff top) there is a point in e^m which is not in the image of f. Retract f away from e^m via this hole. $\square$

Corollary 1.2.6. $\pi_k(\mathbb{S}^n) = 0$ *whenever $k < n$.*

Proof. Consider a map $f : \mathbb{S}^k \to \mathbb{S}^n$. This is homotopic to a cellular map. The only cells in $\mathbb{S}^n$ are itself and a point. $\square$

Corollary 1.2.7. *Let $X^n \subset X$ denote its n-skeleton. Then $X^n \hookrightarrow X$ induces isomorphisms on π_k for $k < n$, and a surjection when $k = n$.*

Proof. Consider $f : \mathbb{S}^k \to X$, cellularly approximate. Surjective whenever $k \leq n$ by lifting the map as in lecture 1. Injective whenever $k + 1 \leq n$ by lifting a homotopy as in lecture 1. (One dimension is lost in injectivity, since we must lift a map from $\mathbb{S}^k \times I$, which adds an additional dimension.) $\square$

Remark. This completes the proof of Whitehead's theorem.

1.2.3 CW-approximation

Definition 1.2.8. Let X be a space. A *CW-approximation* of X is a CW-complex Z and a map $Z \to X$ inducing isomorphisms on homotopy groups. Such a map is called a *weak equivalence.*

Remark. By Whitehead's theorem, on CW-complexes, weak equivalence is equivalent to homotopy equivalence.

Theorem 1.2.9. *Every space X has a CW-approximation $Z \to X$.*

Proof. Build Z inductively. Assume $Z^{n-1} \to X$ has been constructed, inducing isomorphisms on $\pi_k, k < n$. Define

$$Z'^n := Z^{n-1} \vee \bigvee_{\alpha \in \pi_n(X)} \mathbb{S}^n_\alpha.$$

This has a map g into X, since there are maps $Z^{n-1} \to X$ and $\alpha : \mathbb{S}^n_\alpha \to X$. Then form pushout to kill the kernel of g. The universal property of the pushout gives the required map into X. Moreover, gluing copies of $\mathbb{S}^n$ does not change lower homotopy groups. $\square$

1.3 Lecture 3

We have established that CW-approximations exist. However, suppose we have a map $f : X \to Y$ between spaces. If X' and Y' are CW-approximations of X and Y, we don't know if there is an induced map $X' \to Y'$. We prove that this is indeed the case in the following proposition.

Proposition 1.3.1. CW-approximation is functorial up to homotopy. That is, there exists a unique induced map $X' \to Y'$ as described above.

Proof. Suppose $g : Y' \to Y$ is a CW-approximation. Consider the following diagram:

$$\begin{tikzcd}
X' \arrow[rr, "\sim"] & & X \arrow[dl, "\widetilde{f}"] \arrow[dr, "f"] & \\
Y' \arrow[r, "\sim"] & M_g \arrow[rr, "\sim"] & & Y
\end{tikzcd}$$

By a lemma from lecture 2, the map $(X', *) \to (M_g, Y')$ is homotopic to $(X', *) \to (Y', Y')$. □

Corollary 1.3.2. *CW-complexes are unique up to homotopy.*

Proof. Apply the above proposition to $\mathrm{id} : X \to X$. □

1.3.1 Calculating homotopy groups, homotopy excision theorem

Theorem 1.3.3 (Homotopy excision)**.** *Let $X = A \cup_C B$. Assume (A, C) is n-connected, and (B, C) is m-connected. Then the induced map*

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism if $i < n + m$, and surjective if $i = n + m$.

Corollary 1.3.4. *(Freudenthal Suspension)*

$$\pi_i(\mathbb{S}^n) \xrightarrow{\Sigma} \pi_{i+1}(\mathbb{S}^{n+1})$$

is an isomorphism for $i < 2n - 1$, and surjective when $i = 2n - 1$.

Remark. The above is true in general for X $(n-1)$-connected.

Proof. (Freudenthal suspension) Observe that $\Sigma X = CX_+ \cup_X CX_-$, and both sides are homotopic to points. By the homotopy long exact sequence, $\pi_i(X) = \pi_{i+1}(CX_+, X)$, and $\pi_{i+1}(\Sigma X, CX_-) \cong \pi_{i+1}(\Sigma X)$. It remains to understand $\pi_{i+1}(CX_-, X) \to \pi_{i+1}(\Sigma X, CX_-)$. This is an isomorphism for $i + 1 \leq 2n$. □

Proof. (Homotopy excision, outline) By cellular approximation, it is enough to consider $A = C \cup$ cells of dim $\geq n$, $B = C \cup$ cells of dim $\geq m$. Proceed by induction (over cells of A). Let $A = A' \cup e^i$, $X = A \cup_C B = X' \cup e^i$ (where $X' = A' \cup_C B$). This gives a diagram as follows:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & X' \arrow[r] & X
\end{tikzcd}$$

By the inductive hypothesis, the left hand square satisfies the excision theorem. The right hand side is simply adding one cell. We later prove excision for this square. For now, suppose it has already been proven. We have a long exact sequence of homotopy groups from the top and bottom rows of the above diagram:

$$\begin{tikzcd}
\pi_{i+1}(A, A') \arrow[r] \arrow[d, "\cong\text{,prop}"] & \pi_i(A', C) \arrow[r] \arrow[d, "\cong\text{,IA}"] & \pi_i(A, C) \arrow[r] \arrow[d] & \pi_i(A, A') \arrow[r] \arrow[d, "\cong\text{,prop}"] & \pi_{i-1}(A', C) \arrow[d, "\cong\text{,IA}"] \\
\pi_{i+1}(X, X') \arrow[r] & \pi_i(X', B) \arrow[r] & \pi_i(X, B) \arrow[r] & \pi_i(X, X') \arrow[r] & \pi_{i-1}(X', B)
\end{tikzcd}$$

Moreover, the four labelled vertical arrows are isomorphisms by the earlier proposition and by the inductive hypothesis. By the five lemma, the central map is an isomorphism as required.

Next let C be a space, $A = C \cup e^{n+1}$, $B = C \cup e^{m+1}$, $X = A \cup_C B = C \cup e^{n+1} \cup e^{m+1}$. Then $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < m+n$, and a surjection for $i = m+n$. this is a similar proof to earlier proofs in which surjecitivity is first shown, and injectivity follows by a similar argument by lifting homotopies (and hence holds in one lower dimension). Thus only surjectivity is proved. If $b \in (e^{n+1})^\circ, a \in (e^{m+1})^\circ$, then there is a commutative square

$$\begin{tikzcd}
\pi_i(A, C) \arrow[r] \arrow[d, "\cong"] & \pi_i(X, B) \arrow[d, "\cong"] \\
\pi_i(X \setminus \{a\}, X \setminus \{a, b\}) \arrow[r] & \pi_i(X, X \setminus \{b\})
\end{tikzcd}$$

as required.

Next instead of general points a and b, we choose specific useful values. Consider a map $f \in \pi_i(X, B)$; $f : (D^i, D^{i-1}) \to (X, X \setminus \{b\})$. Let $p : D^i \to D^{i-1}$ be a projection. Our new choices of a and b are as follows: let $a' \in e^{m+1}, b' \in e^{n+1}$ with $p(f^{-1}(b')) \cap p(f^{-1}(a')) = \varnothing$. We can find such a' and b' by assuming without loss of generality that everything is smooth. A pair (b', a') does not lie in the image of $f : f^{-1}(e^{n+1}) \times_{D^{i-1}} f^{-1}(e^{n+1}) \to (e^{n+1} \times e^{m+1})$. By smoothness, apply a dimension count: The dimension of the codomain is $n+m+2$, the dimension of the domain is at most $i + i - (i-1) = i + 1$. Thus the map is not surjective, so a' and b' exist.

Finally to complete the proof, we require maps $g : (D^i, D^{i-1}) \to (X \setminus \{a'\}, X \setminus \{a', b'\})$ and $H : (D^i, D^{i-1}) \times I \to (X, X \setminus \{b'\})$, with $g \sim f$ via H. By the choice of a' and b', such maps exist as required. $\square$

1.4 Lecture 4

Corollary 1.4.1. *(of Freudenthal suspension.)* $\pi_n(\mathbb{S}^n) = \mathbb{Z}$.

Proof. It is already known that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. From Freudenthal, the map $\pi_1(\mathbb{S}^1) \to \pi_2(\mathbb{S}^2)$ is cyclic. Since $\pi_2(\mathbb{S}^2)$ contains an element of infinite order (from Homology), the map is an isomorphism. For higher n, Freudenthal immediately gives an isomorphism $\pi_i(\mathbb{S}^i) \to \pi_{i+1}(\mathbb{S}^{i+1})$. $\square$

Example. $\pi_n(\mathbb{S}^1 \vee \mathbb{S}^n) = \mathbb{Z}^\infty$.

1.4.1 Hurewicz theorem

Theorem 1.4.2. *(Hurewicz) Let X be $(n-1)$-connected, $n \geq 2$. Then*

$$\pi_n(X) \cong H_n(X; \mathbb{Z}).$$

Remark. The above theorem holds for relative homology and homotopy.

Proof. Assume without loss of generality that X is a CW-complex, with no cells below dimension n. Then

$$\pi_n(X) \cong \prod_{n\text{-cells of } X} \mathbb{Z} \Big/ \text{relations given by cells} \cong H_n(X; \mathbb{Z}).$$

$\square$

Corollary 1.4.3. *Let $f : X \to Y$ be a map between CW-complexes, ($\pi_1(X), \pi_1(Y)$ abelian). Then f induces an isomorphism on homotopy if and only it induces an isomorphism on homology.*

Proof. Consider $X \hookrightarrow M_f \to Y$. Note that $H_n(M_f, X) = 0$ for all n. Then $\pi_n(M_f, X) = 0$ for all n. $\square$

While an isomorphism between homotopy and homology has been proved, the map is not yet explicit. What is it?

$$\pi_n(X) \to H_n(X), \quad (f : \mathbb{S}^n \to X) \mapsto f_*(0).$$

This is natural up to sign.

Chapter 2

Fibre sequences

2.1 Lecture 4.5

Course is leaving the realm of Hatcher, should be covered effectively in Cohen's preprint.

2.1.1 Basic definitions

Definition 2.1.1. A map $f : E \to B$ has the homotopy lifting property for a space W if, whenever there is a homotopy from W into B, it can be lifted to a homotopy into E.

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\iota"'] & E \arrow[d, "f"] \\
W \times I \arrow[r, "H"] \arrow[ur, dotted, "\exists \widetilde{H}"] & B
\end{tikzcd}$$

- The map f is a *Serre fibration* if it is homotopy lifting with respect to CW-complexes.
- The map f is a *Hurewicz fibration* if it is homotopy lifting with respect to all spaces.

Hereafter, all fibrations are assumed to be Hurewicz fibrations. A *fibre sequence* is $F \hookrightarrow E \to B$ where f is a fibration, and $F = f^{-1}(b_0)$.

Proposition 2.1.2. Pullbacks of fibrations are fibrations.

Proof. The diagram outlines the proof.

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d, "\exists\ (1)"] & E \arrow[d, "f"] \\
W \times I \arrow[r] \arrow[ur, dotted, "\exists\ (2)"] \arrow[urr, dotted, bend right] & B' \arrow[r] & B
\end{tikzcd}$$

Since the homotopy $W \times I \to B' \to B$ lifts to a map $W \times I \to E$, by the universal property of pullbacks, there is a lift of the homotopy $W \times I \to B'$ to E'. □

Example.
- Clearly $\mathrm{id} : E \to E$ and $E \to *$ are fibrations.
- A more instructive example is the projection map $F \times E \to E$. Think fibre bundles.
- If $f : E \to B$ is a fibration, then $f^{-1}(A) \to A$ is also a fibration.

Proposition 2.1.3. Let $A, B \subset X$. Then

$$E(X; A, B) = \{\text{paths in } X, \text{starting in } A \text{ and ending in } B\} \to A \times B$$

is a fibration, where the map is defined by $\gamma \mapsto (\gamma(0), \gamma(1))$.

Proof. The following diagram commutes:

$$\begin{tikzcd}
W \arrow[r, "w \mapsto \gamma_w(-)"] \arrow[d, "w \mapsto (w,0)"] & E(X; A, B) \arrow[d, "\gamma \mapsto (\gamma(0), \gamma(1))"] \\
W \times I \arrow[r, "(w,t) \mapsto (\gamma_A(t), \gamma_B(t))"] & A \times B
\end{tikzcd}$$

Thus $\gamma_A(0) = \gamma_w(0), \gamma_B(0) = \gamma_w(1)$. A lift is constructed by sending

$$(w, t) \mapsto \gamma_A^{-1}(-)|_{[t,0]} + \gamma_w(-) + \gamma_B(-)|_{[0,t]}.$$

□

Remark. Some interesting special cases: If $A = X$ and $B = \{*\}$, we have $E_1 X \to X$: a fibration of paths ending at the base point. The fibres of fibration are exactly loops:

$$\Omega X \hookrightarrow E_1 X \to X.$$

Moreover, the space $E_1 X$ is contractible.

Analogously, one can consider $E_0 X \to X$, a fibration of paths starting at the base point. Then there is again a fibration

$$\Omega X \hookrightarrow E_0 X \to X.$$

We write EX to denote $E_1 X$ (which we use without further mention, by noting that both alternatives give the same theory).

Definition 2.1.4. Let $f : X \to Y$ be any map between topological spaces. The homotopy fibre I_f is the pullback

$$\begin{tikzcd}
I_f \arrow[r] \arrow[d] & EY \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

More explicitly, $I_f = \{(x, \gamma) : \gamma(0) = f(x), \gamma(1) = *\}$.

Remark. $I_f \to Y$ plays the role of being a fibration which "approximates" any f.

2.2 Lecture 5

2.2.1 Basic properties

We eventually wish to construct a fibration

$$\Omega E \to \Omega B \ \to F \to E \to B.$$

Recall our definition of the "homotopy fibre" from the end of the previous lecture: If $f : X \to Y$ is any map, pulling back gives $I_f \to EY$, where I_f is termed the *homotopy fibre*, and $I_f \to X$ is a fibration with fibre ΩY.

Proposition 2.2.1. Let $g : W \to X$ be a map, and $f : X \to Y$ a map.

$$\begin{tikzcd}
 & I_f \arrow[d] & \\
W \arrow[ur, dotted] \arrow[r, "g"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

The map g factors through I_f if and only if $fg \cong *$.

Proof. ($\Rightarrow$) Suppose g factors through I_f. This gives a diagram

$$\begin{tikzcd}
 & I_f \arrow[r] \arrow[d] & EY \cong * \arrow[d, two heads] \\
W \arrow[ur, dotted] \arrow[urr, dotted] \arrow[r, "g"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

($\Leftarrow$) Assume $fg \cong *$, via a homotopy $H : W \times I \to Y$ with $H(-,1) = *, H(-,0) = fg$. Consider the adjoint $\overline{H} : W \to EY$. Pullback gives the required lift. $\square$

Proposition 2.2.2. If $p : X \to Y$ is a fibration, then $p^{-1}(*) \cong I_p$.

Proof. Consider $p^{-1}(*) \to I_p$, sending x to (x, const_*), and $I_p \to p^{-1}(*)$, sending (x,γ) to $\overline{\gamma}(1)$. Consider the following two diagrams:

$$\begin{tikzcd}
* \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\
I \arrow[r, "\gamma"] \arrow[ur, dotted, "E"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
I_p \arrow[r, "(x,\gamma)\mapsto x"] \arrow[d, "\mathrm{ind}_0"] & X \arrow[d, "p"] \\
I_p \times I \arrow[r, "(x,\gamma),t\mapsto\gamma(t)"] \arrow[ur, dotted, "H"] & Y
\end{tikzcd}$$

Define $I_p \to p^{-1}(*)$ as $H(-,1)$. Now $I_p \to p^{-1}(*) \to I_p$ is the end of H given by

$$\begin{tikzcd}
I_p \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{ind}_0"] & I_p \arrow[r] \arrow[d] & X \arrow[d, two heads] \\
I_p \times I \arrow[r] \arrow[ur, dotted, "H"] \arrow[urr, dotted] & EY \arrow[r] & Y.
\end{tikzcd}$$

It remains to prove that the composition $p^{-1}(*) \to I_p \to p^{-1}(*)$ is homotopic to the identity. This is the end of the diagonal map

$$\begin{tikzcd}
p^{-1}(*) \arrow[r] \arrow[d] & X \arrow[d] \\
p^{-1}(*) \times I \arrow[r, "(x,t)\mapsto *"] \arrow[ur, dotted, "H"] & Y.
\end{tikzcd}$$

Since H is a homotopy from the identity we are done. $\square$

Remark. I_p is homotopy equivalent to any $\{(x,\gamma) : \gamma(0) = f(x), \gamma(1) = y\}$, where y is in the path component of $*$. The proposition implies that for any fibration $p : X \to Y$, $p^{-1}(*) \cong p^{-1}(y)$.

Proposition 2.2.3. If we have a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r] \arrow[ur, Rightarrow, "H"] & Y'
\end{tikzcd}$$

which commutes up to homotopy H, then we have a diagram as follows (where the bottom square commutes, and the upper square commutes up to homotopy). The crux of the proof is understanding the horizontal map $I_f \to I_{f'}$:

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d] & \Omega Y' \arrow[d] \\
I_f \arrow[r] \arrow[d] \arrow[ur, Rightarrow, "\widetilde{H}"] & I_{f'} \arrow[d] \\
X \arrow[r] & X'.
\end{tikzcd}$$

Proof. $I_f = \{(x, \gamma) : \gamma(0) = f(x), \gamma(1) = *\}$, and $I_{f'} = \{(x', \gamma') : \gamma'(0) = f'(x'), \gamma'(1) = *\}$. Define the map $I_f \to I_{f'}$ by

$$(x, \gamma) \mapsto (\alpha(x), H(x, 1-t)).$$

Thus the "honest commutativity" is achieved by building the homotopy into the map between I_f and $I_{f'}$. □

Proposition 2.2.4. $\Omega I_f \cong I_{\Omega f}$.

Proof. Apply Ω to the left diagram to obtain the diagram on the right.

$$\begin{tikzcd}
I_f \arrow[r] \arrow[d] & EY \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega I_f \arrow[r, dashed] \arrow[d] & \Omega EY \cong E\Omega Y \arrow[d] \\
\Omega X \arrow[r] & \Omega Y
\end{tikzcd}$$

The top left object in this diagram is the pullback $I_{\Omega f}$, establishing a homotopy (in fact, a homeomorphism). □

Proposition 2.2.5. I If $f_0 \cong f : X \to Y$ via H, then $I_{f_0} \cong I_{f_1}$.

Proof. Left to reader. □

2.3 Lecture 6

Since we have been talking about fibrations for a while, we give a brief excursion into the dual notion.

2.3.1 Cofibrations

Fibrations have the homotopy lifting property:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & E \arrow[d] \\ W \times I \arrow[r] \arrow[ur, dotted, "\exists"] & A \end{tikzcd}$$

Cofibrations have the dual notion of homotopy extension:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & W^I \arrow[d, "\mathrm{ev}_0"] \\ B \arrow[r] \arrow[ur, dotted, "\exists"] & W \end{tikzcd}$$

A *cofibre sequence* is

$$A \hookrightarrow B \to B/A.$$

Example.

- CW pairs are cofibrations.
- $A \hookrightarrow A \vee B, B \hookrightarrow A \vee B$.
- $A \hookrightarrow CA \to \Sigma A$.

We already know pullbacks of fibrations are fibrations. Similarly, pushouts of cofibrations are cofibrations. Moreover, cofibrations satisfy analogous naturality results to fibrations.

Remark. One could state dual notions of propositions concerning fibrations, and prove them for cofibrations.

Proposition 2.3.1. If $f : X \to Y$ is a cofibration, it's an embedding.

Proof. Consider the inclusion $X \hookrightarrow CX$. Since f is a cofibration, there is a lift $\widetilde{H}$ as in the following diagram. We consider the endpoint of this lift:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & CX^I \arrow[d] \\ Y \arrow[r] \arrow[ur, dotted, "\widetilde{H}"] & CX \end{tikzcd}$$

Let $g = H(-, 1)$. Then $g \circ f = i$ which is a homeomorphism onto its image. Thus f is at least an embedding. $\square$

2.3.2 Fibre replacements

Recall that the overarching goal was to understand exact sequences

$$\cdots F \to X \to Y \to 0.$$

More explicitly, we wish to create a long exact sequence by gluing "reasonable" spaces to the left.

Definition 2.3.2. Let $X \to Y$ be a map, and E_f the pullback

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

More explicitly, $E_f = \{(x, \gamma) : f(x) = \gamma(0)\}$.

Proposition 2.3.3. Given $f : X \to Y$, we have $X \xrightarrow{\sim} E_f \xrightarrow{\mathrm{ev}_1} Y$ homotopic to f, with $E_f \to Y$ a fibration. (This will be our notion of a fibre replacement.)

Proof. Consider $X \to E_f$, $x \mapsto (x, \mathrm{const})$. We wish to show that $E_f \to Y$ is a fibration. Consider the following diagram, with $\delta_w(0) = \gamma_w(1)$.

$$\begin{tikzcd} W \arrow[r, "w \mapsto (g(w), \gamma_w)"] \arrow[d] & E_f \arrow[d, "\mathrm{ev}_1"] \\ W \times I \arrow[r, "w, t \mapsto \delta_w(t)"] \arrow[ur, dotted] & Y \end{tikzcd}$$

The lift is given by concatenating paths: $H(w, t) = (g(w), \gamma_w + \delta_w|_{[0,t]})$. The fibre of $E_f \to Y$ is the homotopy fibre $I_f = \{(x, \gamma) : f(x) = \gamma(0), \gamma(1) = *\}$. This gives fibre sequences

$$I_f \to E_f \to Y \to 0, \quad \Omega Y \to I_f \to X.$$

□

Let $f : X \to Y$ be any map. We now begin to construct a "useful" long exact sequence. From above, there is a fibration $\mathrm{pr} : I_f \to X$, with fibre ΩY. But now pr itself has a homotopy fibre, $\lambda : I_{\mathrm{pr}} \to I_f$, with fibre $\lambda^{-1}(*)$.

$$\begin{tikzcd}
& I_{\mathrm{pr}} \arrow[rr, two heads, "\lambda"] & & I_f \arrow[r, two heads, "\mathrm{pr}"] & X \\
\lambda^{-1}(*) \arrow[ur, tail] & & \Omega Y \arrow[ur, tail] & &
\end{tikzcd}$$

Here

$$I_{\mathrm{pr}} = \{((x,\gamma),\delta) \in I_f \times X^I : \gamma(0) = f(x), \gamma(1) = * \in Y, \delta(0) = x, \delta(1) = * \in X\}.$$

The fibre is explicitly given by

$$\lambda^{-1}(*) = \{((*, \mathrm{const}_*), \delta) : \delta(0) = *, \delta(1) = *\}.$$

Thus $\lambda^{-1}(*)$ is just the loops in X! Hence our above diagram becomes

$$\begin{tikzcd}
& I_{\mathrm{pr}} \arrow[r, two heads, "\lambda"] & I_f \arrow[r, two heads, "\mathrm{pr}"] & X \\
\Omega X \arrow[ur, tail] \arrow[r, "\Omega f"] & \Omega Y \arrow[u, "\sim"] \arrow[ur, tail] & &
\end{tikzcd}$$

To extend this further to a juicy long exact sequence, we wish to understand the commutativity of the leftmost triangle. A loop $\gamma \in \Omega Y$ is mapped to $((*, \gamma), \mathrm{const}_*)$. On the other hand, a loop $\delta \in \Omega X$ is sent to $((*, \mathrm{const}_*), \delta)$. Are these homotopic? We must compare $((*, f_*\delta), \mathrm{const}_*)$ and $((*, \mathrm{const}_*), \delta)$.

Consider a loop δ in X. By unlooping, and inspecting the relations in I_{pr}, we find that γ must be δ with reversed parametrisation. More explicitly, we define $H : \Omega X \times I \to I_{\mathrm{pr}}$ by

$$(\gamma, t) \mapsto ((\delta(t), f_*\delta^{-1}|_{[t,0]}), \delta|_{[t,1]}.)$$

This shows that the above diagram can be made commutative up to homotopy if we introduce a sign change! Thus we have a commutative diagram

$$\begin{tikzcd}
& I_{\mathrm{pr}} \arrow[r, two heads, "\lambda"] & I_f \arrow[r, two heads, "\mathrm{pr}"] & X \\
\Omega X \arrow[ur, tail] \arrow[r, "-\Omega f"] & \Omega Y \arrow[u, "\sim"] \arrow[ur, tail] & &
\end{tikzcd}$$

2.4 Lecture 7

2.4.1 Long exact sequence of a fibration

Definition 2.4.1. A sequence of spaces $X_3 \to X_2 \to X_1$ is *exact* at X_2 if $[Z, X_3] \to [Z, X_2] \to [Z, X_1]$ is exact for all Z.

Remark. This means that $\pi_n(X_3) \to \pi_n(X_2) \to \pi_n(X_1)$ is exact for each n.

Lemma 2.4.2. For any map $X \to Y$, the sequence $I_f \to X \to Y$ is exact.

Proof. Exactness means that given a map $Z \to X$, we wish to understand when it arises from a map $Z \to I_f$. This is exactly the lifting property of the diagram

$$\begin{tikzcd} & I_f \arrow[d] & \\ Z \arrow[ur, dotted, "f"] \arrow[r] & X \arrow[r] & Y. \end{tikzcd}$$

But we showed that a lift exists precisely when $Z \to Y$ is homotopy trivial by proposition 2.2.1. □

Lemma 2.4.3. If $X_3 \to X_2 \to X_1$ is exact, then so is $\Omega X_3 \to \Omega X_2 \to \Omega X_1$, and given vertical homotopy equivalences in the following diagram, the second row is also exact.

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d, "\sim"] & X_2 \arrow[r] \arrow[d, "\sim"] & X_1 \arrow[d, "\sim"] \\ Y_3 \arrow[r] & Y_2 \arrow[r] & Y_1 \end{tikzcd}$$

Proof. (Proof for the loop space part of the lemma.)

$$\begin{tikzcd} {[Z, \Omega X_3]} \arrow[r] \arrow[d, "\cong"] & {[Z, \Omega X_2]} \arrow[r] \arrow[d, "\cong"] & {[Z, \Omega X_1]} \arrow[d, "\cong"] \\ {[\Sigma Z, X_3]} \arrow[r] & {[\Sigma Z, X_2]} \arrow[r] & {[\Sigma Z, X_1]} \end{tikzcd}$$

Since the bottom row is exact, the top row is exact. □

Proposition 2.4.4. If $f : X \to Y$ is any map, then

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \to I_f \to X \xrightarrow{f} Y$$

is exact.

Proof. We already know $I_f \to X \to Y$ is exact from an earlier lemma, giving exactness at X. Since ΩY is homotopic to I_λ, where $\lambda : I_f \to X$, we have exactness at I_f. Finally consider the diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "-\Omega f"] \arrow[d, "="] & \Omega Y \arrow[r] \arrow[d, "\sim"] & I_f \arrow[d, "="] \\
\Omega X \arrow[r] \arrow[d, "\sim"] & I_\lambda \arrow[r] \arrow[d, "="] & I_f \arrow[d, "="] \\
I_g \arrow[r] & I_\lambda \arrow[r, "g"] & I_f
\end{tikzcd}$$

to see exactness at ΩY. $\square$

Remark. In fact, $-\Omega f$ has the same image and kernel as Ωf. Therefore we have an exact sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \to I_f \to X \xrightarrow{f} Y.$$

Corollary 2.4.5 (The big theorem!!). *If $F \to X \to Y$ is a fibre sequence, there is a long exact sequence*

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to \Omega F \to \Omega X \to \Omega Y \to F \to X \to Y.$$

Corollary 2.4.6. *If $F \to X \to Y$ is a fibre sequence, then there is a long exact sequence*

$$\cdots \to \pi_2(X) \to \pi_2(Y) \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y).$$

2.4.2 Cofibre analogue: coexact sequences

With cofibrations, we have

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C_f \arrow[r] \arrow[dr] & C_g \arrow[d, "\cong"] \arrow[dr] & \\
& & & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B.
\end{tikzcd}$$

A sequence $A \to B \to C$ is *coexact* if

$$[C, Z] \to [B, Z] \to [A, Z]$$

is exact for each Z. This gives rise to a long coexact sequence

$$A \to B \to C_f \to \Sigma A \to \Sigma B \to \cdots .$$

In particular, given an inclusion $A \hookrightarrow B$, we have

$$A \to B \to B/A \to \Sigma A \to \Sigma B \to \Sigma(B/A) \to \cdots .$$

In fact, because $H^n(X, G) = [X, K(n, G)]$, this gives a long coexact sequence

$$\cdots \to H^{n+1}(B, G) \to H^{n+1}(A, G) \to H^n(B/A, G) \to H^n(B, G) \to H^n(A, G) \to \cdots .$$

Remark. Remarkably, this establishes a duality between homotopy and cohomology, but not between homology and cohomology.

2.4.3 Fibration examples

Remark. If we work with Hurewicz fibrations, we get results in Top. If we work with weak fibrations (Serre fibrations), we get results in CW.

Definition 2.4.7. Let (K, L) be a CW-pair. A map $p : E \to B$ has the homotopy extension property with respect to (K, L) if there is a lift given the following diagram:

$$\begin{tikzcd} K \times \{0\} \cup L \times I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ K \times I \arrow[r] \arrow[ur, dotted, "\exists"] & B \end{tikzcd}$$

Proposition 2.4.8. Let $p : E \to B$ (not pointed). TFAE:

1. p is a weak fibration.
2. p has the homotopy extension property with respect to $(D^n, *)$.
3. p has the homotopy extension property with respect to all CW pairs.
4. If (K, L) is a CW pair, and L is a deformation retract of K, the lift in the following diagram exists:

$$\begin{tikzcd} L \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ K \arrow[r] \arrow[ur, dotted, "\exists"] & B \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2). Extension property with $(I, *)$.

(2) $\Rightarrow$ (3). Notice $(D^n \times I, D^n \times \{0\}) \cong (D^n \times I, D^n \times \{0\} \cup \mathbb{S}^{n-1} \times I)$. Induction over cells of $L \to K$. Assume there is a lift F' for $K \times \{0\} \cup (K, L)^{n-1} \times I$. Extend to $(K, L)^n$. $\square$

2.5 Lecture 8

We first finish proving the proposition from the previous class.

Proposition 2.5.1. Let $p : E \to B$ (not pointed). TFAE:

1. p is an unbased weak fibration.
2. p has the homotopy extension property with respect to $(D^n, *)$.
3. p has the homotopy extension property with respect to all CW pairs.
4. If (K, L) is a CW pair, and L is a deformation retract of K, the lift in the following diagram exists:

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
K \arrow[r] \arrow[ur, dotted, "\exists"] & B
\end{tikzcd}$$

Proof. Last time we proved (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4). Consider $i : L \hookrightarrow K$. Choose a deformation retract r, $H : ir \cong \mathrm{id}_K$. We wish to find a lift in the following diagram on the left. We achieve this by finding a lift in the right diagram as follows:

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d] & E \arrow[d] \\
K \arrow[r, "g"] \arrow[ur, dotted, "\exists"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
K \cup L \times I \arrow[r, "{k \mapsto f(r(k)),\quad (l,t) \mapsto f(l)}"] \arrow[d] & E \arrow[d] \\
K \times I \arrow[r, "{(k,t) \mapsto g(H_t(k))}"] \arrow[ur, dotted, "\widetilde{H}"] & B
\end{tikzcd}$$

(4) $\Rightarrow$ (3). Since $W \to W \times I$ has a deformation retract, we can find the required lift. $\square$

Corollary 2.5.2. *Unbased weak fibrations are based weak fibrations.*

Proof. Let $E \to B$ be a based map, W a based CW complex. Want a based lift as follows:

$$\begin{tikzcd}
W \arrow[r, "\text{based}"] \arrow[d] & E \arrow[d] \\
W \times I \arrow[r, "\text{based}"] \arrow[ur, dotted] & B
\end{tikzcd}$$

But this is the same as finding an unbased lift in the following diagram, by the equivalence in the previous proposition.

$$\begin{tikzcd}
W \cup (* \times I) \arrow[r] \arrow[d] & E \arrow[d] \\
W \times I \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}$$

□

2.5.1 Fibre bundles

Definition 2.5.3. A fibre bundle $p : E \to B$ is a map such that for all $b \in B$ there is a neighbourhood $b \in U \subset B$ such that

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & B \times F_b \arrow[dl] \\
& U &
\end{tikzcd}$$

where $F_b = p^{-1}(b)$.

Remark. All fibres are homeomorphic if B is connected.

Example.
- If F is discrete, then a fibre bundle is just a covering space.
- If $F = [0, 1]$, $[0, 1] \times [-1, 1]/(0, x) \sim (1, -x)$ is the Mobius strip.

Proposition 2.5.4. Fibre bundles are fibrations.

Proof. Suffices to find a lift in the following diagram, by the earlier proposition:

$$\begin{tikzcd}
D^n \arrow[r] \arrow[d] & E \arrow[d] \\
D^n \times I \arrow[r, "H"] \arrow[ur, dotted, "\exists ?"] & B
\end{tikzcd}$$

Consider a cover $B = \bigcup_\alpha U_\alpha$ such that $E|_{U_\alpha} \cong U_\alpha \times F$. Subdivide $D^n = \bigcup_\beta C_\beta$ and $I = \bigcup_\gamma I_\gamma$ such that $H(C_\beta, I_\gamma) \subset U_\alpha$. Inductively lift by inducing on dimension and "order" or cubes. (Here order refers to the natural order of intervals parametrised by time.) Assume there is a lift on ∂C_β and $C_\beta \times \{t\}$ where t is the start of I_γ. Then the following diagram

$$\begin{tikzcd} C_\beta \cup \partial C_\beta \times I \arrow[r, "H_0"] \arrow[d] & U_0 \times F \arrow[d] \\ C_\beta \times I_\gamma \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & U_\alpha \end{tikzcd}$$

has a lift $\widetilde{H}$ given by $(H(x,t), H_0(\text{retraction}(x)))$. $\square$

2.5.2 Fibre bundle examples: low hanging fruit

We now explore some fun examples.

Example.

$$\begin{aligned} \mathbb{S}^0 &\hookrightarrow \mathbb{S}^n \to \mathbb{RP}^n \\ \mathbb{S}^1 &\hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n \\ \mathbb{S}^3 &\hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{HP}^n. \end{aligned}$$

In particular, $\mathbb{CP}^1 \cong \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$, with the map given by $[x_1 : x_2] \mapsto x_1/x_2$. This gives the Hopf fibration

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2.$$

Now the homotopy long exact sequence gives

$$0 = \pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1) = 0.$$

Thus $\pi_3(\mathbb{S}^2) \cong \pi_3(\mathbb{S}^3) = \mathbb{Z}$! This is our first example of computing a non-trivial higher homotopy group. A similar example is given by

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4.$$

Example. There is a fibre bundle

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^\infty \to \mathbb{CP}^\infty.$$

But $\mathbb{S}^\infty$ has no non-trivial homotopy groups, so the long exact sequence of homotopy has isomorphisms

$$\pi_n(\mathbb{CP}^\infty) \cong \pi_{n-1}(\mathbb{S}^1).$$

This shows that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}; 2)$.

Example. There are many Lie group fibre bundles:

$$O(n-1) \hookrightarrow O(n) \to \mathbb{S}^{n-1}$$
$$U(n-1) \hookrightarrow U(n) \to \mathbb{S}^{2n+1}$$
$$Sp(n-1) \hookrightarrow Sp(n) \to \mathbb{S}^{4n+1}.$$

In particular, this gives $\pi_k(O(n-1)) \cong \pi_k(O(n))$ for $k < n-2$.

Example. More generally, let $G_n(\mathbb{R}^k)$ be the n-planes in $\mathbb{R}^k$, i.e. the Grassmannian $G(n,k)$. Let $V_n(\mathbb{R}^k) = V(n,k)$ be the orthonormal sets of size n in $\mathbb{R}^k$. Then there is a fibre bundle

$$O(n) \hookrightarrow V(n,k) \to G(n,k).$$

More on this example in the next class!

2.6 Lecture 9

We continue the fibre bundle example from the previous class.

Example. Consider

$$O(n) \to V(n,k) \to G(n,k).$$

Suppose $n < m < k$. Then there is a fibration

$$V(m-n, k-n) \to V(m,k) \to V(n,k).$$

For $n = 1$, this is really just

$$V(m-1, k-1) \to V(m,k) \to \mathbb{S}^{k-1}.$$

By induction, it follows that $V(m,k)$ is $(k-m-1)$-connected. In particular $V(m,\infty)$ is weakly contractible. But now from the sequence

$$O(n) \to V(n,k) \to G(n,k),$$

we have

$$\pi_n(G(m,\infty)) \cong \pi_{n-1}(O(m)), \quad \text{so that } G(m,\infty) \cong BO(m).$$

Chapter 3

Cohomlogy

Notice: $\widetilde{H}^n(-,G) : \text{Top} \to \text{Ab}$ is contravariant. Is it representable as $\widetilde{H}^n(-,G) \cong [-,K(G,n)]$? The goal of this chapter is answer this question, which develops as an analogue to the observation that π_n are representable covariant functors. For the next week or so we establish that reduced homology is representable.

Remark. Observe that reduced homology is necessary. If reduced homology isn't used, then we require $[*,K(G,n)]$ to be non-trivial, which is certainly impossible.

3.1 Construction of $K(G,n)$

Remark. Assume $K(G,n)$ exists. What is its homotopy type? We know that

$$\pi_k(K(G,n)) \cong [\mathbb{S}^k, K(G,n)] \cong \widetilde{H}^n(\mathbb{S}^k, G) = \begin{cases} G & k = n \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 3.1.1. Suppose (X,A) is an r-connected CW pair and A is s-connected. Then

$$\pi_i(X,A) \xrightarrow{\cong} \pi_i(X/A)$$

for $i \leq r+s$, and a surjection when $i = r+s+1$.

Proof. Homotopy excision. Observe that, in the correct range,

$$\pi_i(X,A) \cong \pi_i(X \cup CA, CA) \cong \pi_i(X \cup CA) \cong \pi_i(X/A).$$

□

We now construct $K(G,n)$s. We do this skeleton-wise:

$$(K(G,n))^{n+1} = \bigvee_{\alpha \in G} \mathbb{S}^n_\alpha \cup \bigvee_{\beta \text{ relation in } G} D^{n+1}_\beta.$$

Consider the pair $((K(G,n)^{n+1}), \bigvee_{\alpha\in G}\mathbb{S}^n_\alpha)$. By the above lemma we then have an exact sequence

$$\begin{tikzcd}
\pi_{n+1}(K(G,n)^{n+1}, \bigvee_{\alpha\in G}\mathbb{S}^n) \arrow[r] \arrow[d, "\cong"] & \pi_n(\bigvee \mathbb{S}^n) \arrow[r] \arrow[d, "\cong"] & \pi_n(K(G,n)^{n+1}) \arrow[r] \arrow[d, "\cong"] & 0 \\
\pi_{n+1}(K(G,n)^{n+1}/\bigvee \mathbb{S}^n) & \bigoplus_{\alpha\in G}\mathbb{Z} & G. &
\end{tikzcd}$$

By gluing cells to destroy only higher homotopies, this gives a construction.

Proposition 3.1.2. The construction above is natural, i.e. if X has the desired homotopy type, then there exists a weak equivalence $K(G,n)\to X$.

Proof. Constructed cell-wise. The crux of the proof is that if $Y=\bigvee\mathbb{S}^n\cup e^{n+1}$ and there is a map $f:\pi_n(Y)\to\pi_n(X)$, then f is induced from a map $Y\to X$. The proof of the crux of the proof is that $\pi_n(X)$ gives a map $\bigvee\mathbb{S}^n\to X$. Extend this to e^{n+1} by

$$e^{n+1}\to\bigvee\mathbb{S}^n\to\bigvee\mathbb{S}^n\cup e^{n+1}\xrightarrow{\text{exists}} X.$$

This gives $K(G,n)^{n+1}\to X$ as required, with maps of higher cells being trivial. □

Remark. Observe that $\pi_i(\Omega K(G,n))\cong\pi_{i+1}(K(G,n))$. This gives a weak equivalence between $K(G,n)$ and $\Omega K(G,n+1)$. By adjointness we also have a weak equivalence $\Sigma K(G,n)\to K(G,n+1)$.

To show that cohomology is representable, it will be shown that

$$\widetilde{H}^n(X,G)\cong[X,K(G,n)]$$

by showing that $[-,K(G,n)]$ is a generalised cohomology theory; a contravariant functor from CW to Ab.

Definition 3.1.3. A reduced generalised cohomology theory is a collection of functors h^n (from $\mathrm{CW}^{\mathrm{op}}$ to Ab) that

1. takes homotopy equivalences to isomorphisms,
2. takes a cofibre sequence $A\to B\to B/A$ to an exact sequence

$$\cdots\to h^n(B/A)\to h^n(B)\to h^n(A)\to h^{n-1}(B/A)\to\cdots.$$

3. and satisfies $h^n(\bigvee_\alpha X_\alpha)=\prod_\alpha h^n(X_\alpha)$.

Remark. The second condition is equivalent to requiring that each $h^n(B/A)\to h^n(B)\to h^n(A)$, and $h^n(\Sigma A)=h^{n+1}(A)$.

3.2 Lecture 10

Recall from the previous lecture that a cohomology theory is a collection of functors h^* which (i) sends homotopy equivalences to isomorphisms, (ii) cofibre sequences to long exact sequences, and is (iii) trivial on points.

Remark. The contents of the lectures are back into Hatcher.

Example. Consider $[-, K(G,n)]$. It is easy to see that this satisfies properties (i) and (iii) for being a cohomology theory. Moreover, to see that it satisfies the second condition, consider a cofibre sequence $B/A \to B \to A$. Then it can be verified that

$$\cdots \to [B/A, K(G,n)] \to [B, K(G,n)] \to [A, K(G,n)] \to [B/A, K(G,n-1)] \to \cdots .$$

is exact, since

$$[B/A, K(G,n-1)] \cong [B/A, \Omega K(G,n)] \cong [\Sigma B/A, K(G,n)].$$

In general, any any collection (X_n) with $\Omega X_n \cong X_{n-1}$ gives a cohomology theory $[-, X_n]$, called the Ω spectrum.

3.2.1 Brown Representability

Theorem 3.2.1. *Every reduced cohomology theory has the form $h^n(-) = [-, K_n]$ for some Ω-spectrum K_n, which is unique up to homotopy.*

Corollary 3.2.2. $H^n(-, G) \cong [-, K(G,n)]$.

Proof. Cohomology must be represented by something. Applying H^n to spheres shows that the spaces must be $K(G,n)$. □

We now organise the preparations for the proof, which will take a lecture and a half.

- We need only focus on one level of $h : \mathrm{CW}_* \to \mathrm{Ab}$, since we can glue together the rest.
- An alternative condition for (ii) is Mayer-Vietoris: given A and B, $h(A \cup B) \to h(A) \oplus h(B) \to h(A, B)$ is exact. Thus the two properties can be used interchangeably.

It is well known that (ii) implies Mayer-Vietoris. For the converse, suppose $A \to B$ is a cofibration. The following diagram shows that (ii) must hold:

$$\begin{tikzcd}
h(B \cup CA) \arrow[r] \arrow[d, "\cong"] & h(B) \oplus h(CA) \arrow[r] \arrow[d, "\cong"] & h(B \cap CA) \arrow[d, "\cong"] \\
b(B/A) \arrow[r] & h(B) \arrow[r] & h(A)
\end{tikzcd}$$

Proposition 3.2.3. Let $h : \mathrm{CW}_* \to \mathrm{Ab}$ satisfy (i), MV, and (iii). Then there is a $(K, u), u \in h(K)$ such that $[X, K] \to h(X), f \mapsto f^*(u)$ is a bijection.

Remark. (K, u) is called a *universal pair* for h. If (K, u) and (K', u') are universal pairs, then $[K, K'] = h(K)$, with $k \mapsto k'$. Universality comes for free.

Remark. (K, u) is called *n-universal* if $\pi_i(K) \to h(S^i)$ is bijective for $i < n$ and surjective when $i = n$. (K, u) is $*$-universal if it is n-universal for all n.

Lemma 3.2.4. Given any (Z, z) where $z \in h(Z)$, Z connected CW. There is a $*$-universal (K, u), $Z \subset K$ and $z = u|_h(Z)$.

Proof. Proof by induction: Let $K_1 = z \vee \bigvee_\alpha \mathbb{S}^1_\alpha$ for $\alpha \in h(\mathbb{S}^1)$. Set u by $u|_{h(Z)} = z$ and $u|_{h(\mathbb{S}^1_\alpha)} = \alpha$. Then $[\mathbb{S}^1, K_1] \to h(\mathbb{S}^1)$ is surjective. For the inductive step, suppose $\pi_n(K_n) \to h(\mathbb{S}^n)$ is surjective. Let $\alpha \in \ker$, and consider $f : \bigvee_\alpha \mathbb{S}^n_\alpha \to K_n$. This gives a cofibre sequence

$$\bigvee \mathbb{S}^1_\alpha \to M_f \to C_f =: K'_{n+1}.$$

By property (ii), this gives an exact sequence

$$h(K'_{n+1}) \to h(K_n) \to \bigoplus h(\mathbb{S}^n).$$

Now define $K_{n+1} = K'_{n+1} \vee \bigvee_\beta \mathbb{S}^1_\beta, \beta \in h(\mathbb{S}^{n+1})$. Once can verify that K_{n+1} satisfies the required properties.

Finally to define the $*$-universal pair (K, u), define $K = \bigcup K_n \times [n, n+1]$. Let A be the union of even indices, and B the odd indices. Since A and B are disjoint unions, one can define $u_a, u_b \in h(A), h(B)$. Then $K = A \cup B$, so MV can be used to glue u_a and u_b as required. One can show that (K, u) is now $*$-universal. □

Lemma 3.2.5. Let (A, a) and (X, x) be universal pairs. Assume $A \to X$ is a cofibration with $x|_{h(A)} = a$. Then there exists a lift into a $*$-universal (K, u):

$$\begin{tikzcd}
(A,a) \arrow[r] \arrow[d] & (K,u) \\
(X,x) \arrow[ur, dotted, "\exists"] &
\end{tikzcd}$$

The proof is left as an exercise for the reader, since my live-TeXing was too slow to write up the proof outline at the end of the lecture.

3.3 Lecture 11

3.3.1 Finishing the proof of Brown Representability

We complete the proof of Brown representability. Recall that we want to find universal (K, u) such that $[X, K] \cong h(X)$, $f \mapsto f^*u$. In the previous lecture we proved that we can find $*$-universal (K, u), i.e. $[\mathbb{S}^n, K] \cong h(\mathbb{S}^n)$ for each n. We ended the lecture with a lemma,

Lemma 3.3.1. Let (A, a) and (X, x) be universal pairs. Assume $A \to X$ is a cofibration with $x|_{h(A)} = a$. Then there exists a lift into a $*$-universal (K, u):

$$\begin{tikzcd} (A,a) \arrow[r] \arrow[d] & (K,u) \\ (X,x) \arrow[ur, dotted, "\exists"] & \end{tikzcd}$$

In fact, using this lemma we can prove that $*$-universal pair is universal. That is, if $[X, K] \cong h(X)$ for all X an n-sphere, then it holds for all X.

Proof. First we show that $[X, K] \cong h(X)$ is surjective. Take $A = *$ in the lemma. Then the existence of the lift proves surjectivity. For injectivity, consider $f_1 \vee f_2 : (X \wedge \partial\partial I, x) \to (K, u)$. Then by the previous lemma, since $(X \wedge \partial\partial I, x) \hookrightarrow (X \wedge I, x)$, we have a homotopy $f_1 \sim f_2$. This gives injectivity. □

In summary,

Proposition 3.3.2. There exists a universal pair (K, u) for h. That is, if $h : \mathrm{CW} \to \mathrm{Ab}$ satisfies MV and property (iii) from the previous lecture, then there exists (K, u) such that $[X, K] \cong h(X)$, $f \mapsto f^*u$.

Theorem 3.3.3. *Every reduced cohomology theory has the form $h^n(-) = [-, K_n]$ for some Ω-spectrum K_n, which is unique up to homotopy.*

Proof. Let $h^\bullet$ be a reduced cohomology theory. For each h^n, we can find universal pairs (K_n, u). Next we show that $K_n \cong \Omega K_{n+1}$. But we already know that

$$[K_n, \Omega K_{n+1}] \cong [\Sigma K_n, K_{n+1}] \cong h^{n+1}(\Sigma K_n) \cong h^n(K_n) \cong [K_n, K_n].$$

Thus if ω is the image of $\mathrm{id} \in [K_n, K_n]$ lying in $[K_n, \Omega K_{n+1}]$, then ω is an isomorphism between K_n and ΩK_{n+1}. Finally it remains to show that $[X, \Omega K_{n+1}] \to h^n(X)$ is in fact a group homomorphism. This comes from a cogroup structure and is left as an exercise. □

Example. The map $H^1(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, $\alpha \mapsto \alpha^2$ is trivial. Proof: $\alpha = f^*u$ for some $u \in K(\mathbb{Z},1) = \mathbb{S}^1$. Then $u^2 = 0$, so $\alpha^2 = 0$.

The map $H^1(X,\mathbb{Z}/2\mathbb{Z}) \to H^2(X,\mathbb{Z}/2\mathbb{Z})$ defined analogously may not be trivial, since $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^\infty$.

If R is a ring, let $K_n = K(R,n)$, $f \in H^n(X,R)$, $g \in H^m(Y,R)$. This gives maps

$$X \times Y \to K_m \times K_n \to K_n \wedge K_m \xrightarrow{\mu} K_{m+n}.$$

(where we must still find μ). We observe that (by the Kunneth formula for reduced homology)

$$H_{m+n}(K_n \wedge K_m) \cong H_n(K_n) \otimes H_m(K_m) \cong R \otimes R.$$

Then

$$[K_n \wedge K_m, K_{m+n}] \cong H^m(K_n \wedge K_m, R) \cong \hom(H_{n+m}(K_n \wedge K_m), R) \cong \hom(R \otimes R, R) \ni \mu.$$

3.3.2 Homology, spectra, and stable homotopy

Definition 3.3.4. A homology theory is a co-cohomology theory. Explicitly, it is a collection of covariant functors $h_n : \mathrm{CW} \to \mathrm{Ab}$ such that

(i) h_n is homotopy invariant.

(ii) If $B \to A \to A/B$ is a cofibre sequence, then it gives rise to a long exact sequence of homology.

(iii) $h_n(\bigvee_\alpha X_\alpha) = \prod_\alpha h_n(X_\alpha)$.

Example. Homotopy is not a homology theory.

Example. Stable homotopy is a homology theory! That is,

$$\pi_n^S(X) := \lim_{\to} \pi_{i+n}(\Sigma^i X),$$

with connecting maps

$$[\mathbb{S}^{i+n}, \Sigma^i X] \to [\Sigma\mathbb{S}^{i+n}, \Sigma\Sigma^i X],$$

is a homology theory. It is easy to see that stable homotopy is homotopy invariant. For part (ii), observe that $\pi_n^S(X) \cong \pi_{n+1}^S(\Sigma X)$, so it remains to prove that whenever $B \to A \to A/B$ is a cofibre sequence, then there is an exact sequence $\pi_n(B) \to \pi_n(A) \to \pi_n(A/B)$. We already know that there is an isomorphism $\pi_n(A,B) \cong \pi_n(A,B)$ for A, B highly connected (which occurs in the direct limit since it works approximately when $i \cong n$). This proves (ii). Finally for (iii), we make the same argument but on crack. (Literal quote from lecturer.) Observe that

$$\pi_{n+i}(\Sigma^i X) \times \pi_{n+i}(\Sigma^i Y) \cong \pi_{n+i}(\Sigma^i X \times \Sigma^i Y) \cong \pi_{n+i}(\Sigma^i X \vee \Sigma^i Y).$$

To be precise, the right hand side is the $2i-1$ skeleton of $\Sigma^i X \times \Sigma^i Y$, so the isomorphism on right holds approximately for order $2i$ and above. This proves (iii), so stable homotopy is a homology theory as desired.

In the next lecture we will generalise this idea, and associate a homology theory to every spectrum.

Chapter 4

Obstruction theory

4.1 Lecture 12

Late to class (since I was talking to the police...), so the notes start briefly:

Generalisation: Let (K_n) be a spectrum. Then $(X \wedge K_n)_n$ is also a spectrum. The homotopy groups of the spectrum is given by $\pi_n K = \lim_i \pi_{i+n}(K_i)$.

Claim: $\pi_*(-\wedge K)$ is a homology theory for all spectra K. This was proved, but I missed the proof due to lateness.

Example.

- Using spheres gives stable homotopy theory.
- Using $K = K(G, n)$ gives homology theory.

4.1.1 Postnikov towers (with a view to obstruction theory)

Motivation: Maps out of CW complex X can be defined skeletonwise. What about maps into CW complexes? This requires data of the form

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_2 \arrow[d] \\
Y \arrow[r] \arrow[ur] \arrow[uur] & X_1
\end{tikzcd}$$

where $X = X_\infty$. Thus we want a construction where X is described as an inverse limit.

Let X be a CW complex. Then we can find a *Postnikov tower* of X, namely

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & X_1 \end{tikzcd}$$

where $\pi_i(X_k) = \begin{cases} \pi_i(X) & i \le k \\ 0 & i > k. \end{cases}$ By fibre replacement, it can be assumed that each $X_{k+1} \to X_k$ is a fibration. The fibre is $K(\pi_{k+1}(X), n)$.

Proposition 4.1.1. For an arbitrary sequence of fibrations $\cdots \to X_{k+1} \to X_k \to X_{k-1} \to \cdots$, the map $\lambda : \pi_i \lim_{\leftarrow} X_k \to \lim_{\leftarrow} \pi_i X_k$ is surjective for all i, and injective if $\pi_{i+1}(X_k) \to \pi_{i+1}(X_{k-1})$ is surjective for large k.

Corollary 4.1.2. *$X = X_\infty$ in the Postnikov tower.*

Proof. For surjectivity, let $f_k : \mathbb{S}^i \to X_k$ be in $\pi_i(X_k)$ such that $[\rho_* f_{k+1}] = [f_k]$. Consider the following diagram, and replace f_{k+1} with the end of it (in the homotopy):

$$\begin{tikzcd} \mathbb{S}^i \arrow[r, "f_{k+1}"] \arrow[d, "\iota_0"] & X_{k+1} \arrow[d, "\rho"] \\ \mathbb{S}^i \times I \arrow[r, "\rho_* f_{k+1} \sim f_k"] \arrow[ur, dotted, "H"] & X_k \end{tikzcd}$$

For injectivity, assume $\pi_i(X_{k+1}) \to \pi_i(X_k)$ is surjective for all k. Let $f : \mathbb{S}^i \to \lim X_k$ be in the kernel of λ. Consider $f_k : \mathbb{S}^1 \to X_k$ for each k with $\rho_* f_{k+1} = f_k$, and $F_k : D^{i+1} \to X_k$ with $F_k|_{\mathbb{S}^i} = f_k$. Then $\rho_* F_k$ and F_k can be glued together to get $g_k : \mathbb{S}^{i+1} \to X_k$ (since two copies of the disk glue to give the sphere). Since $X_{k+1} \to X_k$ induces a surjective map on homotopies, $g_k : \mathbb{S}^{i+1} \to X_k$ lifts to a map into X_{k+1}. Thus we could have chosen F_{k+1} such that $\rho_* F_{k+1} = F_k$ rel $\mathbb{S}^i$. This gives $F : D^{i+1} \to \lim_{\leftarrow} X_k$ as required. $\square$

Corollary 4.1.3. *In a Postnikov tower of fibrations, $X \cong \lim_{\leftarrow} X_k$.*

Definition 4.1.4. A principal fibration is a fibration of the form

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "\cong"] & B \arrow[d, "\cong"] & \\ \Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r] & B'. \end{tikzcd}$$

Assume every fibration in a Postnikov tower is principal. Then the entire Postnikov tower can be recovered by knowing the maps $[X_k, K(\pi_{k+1}(X), k+2)] = H^{k+2}(X_k, \pi_{k+1}(X))$. In summary:

Proposition 4.1.5. Obstructions to lifting are just $H^{k+2}(X_k, \pi_{k+1}(X))$! We work to formalise this some more, and clear up the assumption of principality of Postnikov towers.

4.2 Lecture 13

Recall that the theme of the current lectures is to determine when we can lift maps up Postnikov towers:

$$\begin{tikzcd}
 & \vdots \arrow[d, two heads] & \\
K(\pi_2(X), 2) \arrow[r] & X_2 \arrow[d, two heads] & \\
Y \arrow[r] \arrow[ur, dotted, "?"] \arrow[rr, bend right, "f"'] & X_1 \arrow[r] & K(\pi_2(X), 3)
\end{tikzcd}$$

We claim that the lift ? exists if the composition f is trivial in $[Y, K(\pi_2(X), 3)] = H^3(Y, \pi_2(X))$.

Theorem 4.2.1. *A connected CW-complex X has a principal Postnikov tower iff the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for $n > 1$.*

Proof. We first prove ($\Rightarrow$). Suppose $F \to E \to B$ is principal (with $A \to X$ lying over $F \to E$). The fibre of $A \to X$ is the loop space ΩB. The action of $\pi_1(A)$ on $\pi_n(X, A)$ is the same as the action of $\pi_1(F)$ on $\pi_n(E, F)$. There is a map $\pi_n(E, F) \to \pi_n(B)$, which is in fact an isomorphism by observing that they sit in the same long exact sequence. Observe that the action of $\pi_1(F)$ on $\pi_n(E, F)$ becomes trivial in $\pi_n(B)$. It follows that $\pi_1(X)$ acts trivially on $\pi_n(X_n, X_{n+1})$. Then the first map in $\pi_k(X_{n+1}) \to \pi_k(X_n) \to \pi_k(X_{n+1}, X_n)$ is an isomorphism except when $k = n+2$, which gives the result. $\square$

For the converse, we begin with a lemma.

Lemma 4.2.2. Let (X, A) be a CW-pair, with X, A both connected. Then the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\sim"] & X \arrow[d] \arrow[dr] & \\
F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

is a principal fibration if and only if $\pi_1(A)$ acts trivially on $\pi_n(X, A)$ for all $n > 1$. Here we assume the homotopy fibre of $A \to X$ is $K(G, n)$.

Proof. We need only show ($\Leftarrow$). Let $\pi_{n+1}(X, A)$ be the first non-trivial homotopy. Then it is isomorphic to the $n+1$th homology, by Hurewicz. Then

$$\pi_{n+1}(X, A) \cong H_{n+1}(X, A) \cong H_{n+1}(X/A) \cong \pi_{n+1}(X/A) \cong G.$$

Consider the map $X \to X/A$. Add higher cells to X/A to kill all homotopy groups above $n+1$, which gives a $K(G, n+1)$. Using a fibre replacement, we obtain the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d, "\sim"] \arrow[dr] & X/A \arrow[d] \\
F \arrow[r] & E \arrow[r] & K(G, n+1)
\end{tikzcd}$$

The map $A \to F$ is a weak equivalence by the 5-lemma (since the left side of the diagram can be extended on with $K(G, n)$s). $\square$

Proof. We complete the proof of the earlier theorem. ($\Rightarrow$) was already complete. By the above lemma, the actions $\pi_1(X)$ on $\pi_k(X_n, X_{n+1})$ are all trivial as required. $\square$

4.2.1 Moore-Postnikov towers

A more general notion is that of a Moore-Postnikov tower. I guess it's even more of a Postnikov tower than Postnikov towers themselves.

Definition 4.2.3. Let $f : X \to Y$ be a map between CW-complexes. A Moore-Postnikov tower is

$$\begin{tikzcd}
& \vdots \arrow[d] \arrow[ddr] & \\
& Z_2 \arrow[d] \arrow[dr] & \\
X \arrow[uur] \arrow[ur] \arrow[r] \arrow[rr, bend right, "f"'] & Z_1 \arrow[r] & Y
\end{tikzcd}$$

satisfying

- $\pi_i(X) \to \pi_i(Z_n)$ is an isomorphism for $i < n$, and surjective when $i = n$.

- $\pi_i(Z_n) \to \pi_i(Y)$ is an isomorphism for $i > n$ and injective when $i = n$.
- The tower is made up of fibrations in the sense that

$$K(\pi_n(\text{homotopy fibre } f), n) \to Z_{n+1} \to Z_n$$

  are fibrations. This is unique up to homotopy.

Theorem 4.2.4. *Every map $X \to Y$ has a Moore-Postnikov tower. If $\pi_1(X)$ acts on $\pi_n(M_f, X)$, then this is principal.*

Proof. The construction is as expected: It can be built up to satisfy the first two properties (but not th fibration property) by gluing cells inductively. To ensure that the third property is satisfied, we use fibre replacements as in the following diagram:

$$\begin{tikzcd}
 & X_2' \arrow[r, "\sim"] \arrow[d] & X_2 \arrow[dl] \arrow[dr, "\text{composition}"] & \\
X \arrow[ur] \arrow[r] & X_1' \arrow[rr] & & Y
\end{tikzcd}$$

Next we use a giant five lemma thing to investigate the homotopy fibres:

$$\begin{tikzcd}
\pi_{i+1}(Z_{n+1}) \arrow[r] \arrow[d, "="] & \pi_{i+1}(Z_n) \arrow[r] \arrow[d] & \pi_{i+1}(Z_n, Z_{n+1}) \arrow[r] \arrow[d] & \pi_i(Z_{n+1}) \arrow[r] \arrow[d, "="] & \pi_i(Z_n) \arrow[d] \\
\pi_{i+1}(Z_{n+1}) \arrow[r] & \pi_{i+1}(Y) \arrow[r] & \pi_{i+1}(Y, Z_{n+1}) \arrow[r] & \pi_i(Z_{n+1}) \arrow[r] & \pi_i(Y) \\
\pi_{i+1}(X) \arrow[r] \arrow[u] & \pi_{i+1}(Y) \arrow[r] \arrow[u, "="] & \pi_{i+1}(Y, X) \arrow[r] \arrow[u] & \pi_i(X) \arrow[r] \arrow[u] & \pi_i(Y) \arrow[u, "="]
\end{tikzcd}$$

By two applications of the five lemma, we have $\pi_{n+1}(Z_n, Z_{n+1}) \cong \pi_{n+1}(X, Y)$. Other π_is vanish. By the long exact sequence of homotopy fibres and long exact sequence of pairs, we have

$$\pi_{n+1}(X', Y) \cong \pi_n(\pi_n(\text{homotopy fibre } f)).$$

Principality follows from the same reason as earlier. □

4.3 Lecture 14

4.3.1 Obstruction theory

Often we want to lift a map as in either of the following diagrams:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \\
W \arrow[ur, dotted]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Y \arrow[d, two heads] \\
W \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}
$$

The left diagram we refer to as case 1, the right diagram as case 2.

Case 1. Assume X has a principal Postnikov tower. Then there is a diagram as follows:

$$
\begin{tikzcd}
 & & \vdots \arrow[d, two heads] & \\
 & & X_2 \arrow[r] \arrow[d, two heads] & K(\pi_3(X), 4) \\
A \arrow[r] \arrow[d, hook] & X \arrow[r] & X_1 \arrow[r] \arrow[d, two heads] & K(\pi_2(X), 3) \\
W \arrow[rr] & & * = X_0 \arrow[r] & K(\pi_1(X), 2)
\end{tikzcd}
$$

We wish to find a lift $W \to X$. Assume we already have the following diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X_{n+1} \arrow[d] & \text{homotopy fibre}(f) \arrow[l] \\
W \arrow[r] \arrow[rr, bend right] & X_n \arrow[r, "f"] & K(\pi_{n+1}(X), n+2)
\end{tikzcd}
$$

Then we have a lift if $\omega_n : W \cup CA \to K(\pi_{n+1}(X), n+2)$ is trivial, i.e. if $\omega_n \in H^{n+2}(W, A; \pi_{n+1}(X))$ is trivial.

Corollary 4.3.1. *If X is abelian and $H^{n+1}(W, A; \pi_n(X)) = 0$, then the extension as in case 1 is guaranteed to exist.*

Now we consider the general case of the lifting problem, as in the following diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\
W \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}
$$

This time we use Moore-Postnikov towers. Assume a principal Moore-Postnikov tower exists, and consider the following diagram, where F denotes the homotopy fibre of the map $X \to Y$:

$$\begin{tikzcd}
 & & \vdots \arrow[d, two heads] & \\
 & & Z_2 \arrow[r, two heads] \arrow[d, two heads] & K(\pi_3(F), 4) \\
A \arrow[r] \arrow[d, hook] & X \arrow[r] & Z_1 \arrow[r, two heads] \arrow[d, two heads] & K(\pi_2(F), 3) \\
W \arrow[rr] \arrow[urr, dotted] & & y = Z_0 \arrow[r, two heads] & K(\pi_1(F), 2)
\end{tikzcd}$$

Lift by covering space theory, and obtain the same conclusion as earlier:

Proposition 4.3.2. There is a lift $W \to X$ as in the previous diagram if $\omega \in H^{n+1}(W, A; \pi_n(F))$ is trivial.

4.3.2 Dold-Thom theorem

Definition 4.3.3. Let X be a space, and define $SP_n(X) = \prod_n X/S_n$. (This is the symmetric product, modding out by the symmetric group.) Observe that there is a map $SP_n(X) \to SP_{n+1}(X)$, sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, *)$.

Definition 4.3.4. The bigg symmetric product is $SP(X) := \lim_{\to} SP_n(X)$. SP is a homotopy functor (from Top to Top).

Example. $SP(\mathbb{S}^2) \cong \mathbb{CP}^\infty$. The right side can be identified with all non-zero expressions $a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ up to scaling of coefficients. On the left, observe that there is a canonical map $(\mathbb{S}^2)^\infty \to SP(\mathbb{S}^2)$. The map $(\mathbb{S}^2)^\infty \to \mathbb{CP}^\infty$ sending $(b_1, \ldots, b_n) \to (z - b_1) \cdots (z - b_n)$ induces the claimed homeomorphism.

Theorem 4.3.5. *The functor $X \mapsto \pi_i SP(X)$, for $i \geq 1$, coincides with the functor $X \mapsto H_i(X, \mathbb{Z})$ on the category of connected CW complexes. Wow!*

Example.

- $SP(\mathbb{S}^n) = K(\mathbb{Z}, n)$.
- If M denotes the *Moore space*, then $SP(M(G, n)) = K(G, n)$.

Proof strategy:

1. Want to show that $X \mapsto \pi_* SP(X)$ is a homology theory, h_i.

2. Show that $h_i(\mathbb{S}^2) = \pi_i(\mathbb{CP}^\infty) = H_i(\mathbb{S}^2; \mathbb{Z})$.

3. Profit.

Therefore most of the work lies in proving that $X \mapsto \pi_* SP(X)$ is a homology theory! This is a very structured process, good news.

Proposition 4.3.6. $X \mapsto h_*(X) := \pi_* SP(X)$ is a homology theory, where X are connected CW complexes.

Proof. (i) Homotopy invariance is immediate.

(iii) Want to show that $h_i(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha h_i(X_\alpha)$. To do this, write

$$SP\Big(\bigvee_\alpha X_\alpha\Big) = \prod_\alpha SP(X_\alpha).$$

This follows from a counting argument, where the right side can be expressed as a union of finitely many functors. The real meat of the proof is in proving the homology long exact sequence.

(ii) We need an exact sequence

$$\cdots \to \pi_i(SP(A)) \to \pi_i(SP(X)) \to \pi_i(SP(X/A)) \to \pi_{i-1}(SP(A)) \to \cdots.$$

This would follow if $SP(A) \to SP(X) \to SP(X/A)$ were a fibration! But unfortunately, it isn't. However the result still holds if $SP(A)$ is homotopic to the homotopy fibre of $SP(X) \to SP(X/A)$. This is tough! Requires the notion of quasi-fibrations, and their local-to-global properties. □

Definition 4.3.7. $f : E \to B$ is called a *quasi-fibration* if the fibre of a point b is homotopic to the homotopy fibre of $E \to B$.

Lemma 4.3.8. $f : E \to B$ is a quasi-fibration if and only if $\pi_i(E, f^{-1}(b)) \cong \pi_i(B)$ for all b.

Proof. They are in the same long exact sequence if and only if the above condition holds. □

The next lemma is juicy, and will be proven in the next lecture.

Lemma 4.3.9. $f : E \to B$ is a quasi-fibration if any of the following hold:

1. $B = V_1 \cup V_2$, where each $f^{-1}(V_i) \to V_i$, and $f^{-1}(V_1 \cap V_2) \to V_1 \cap V_2$ is a quasi-fibration.

2. $B = B_1 \cup B_2 \cup \cdots$, where $B_i \subset B_{i+1}$, and every compact $K \subset B$ is supported in $K \subset B_k$ for some k, with each $f^{-1}(V_i) \to V_i$ a quasi-fibration.

3. There is a deformation retract F_t of E into E' covering $F_t : B \to B'$ such that $E' \to B'$ is a quasi-fibration, and the fibres of b and $F(b)$ are weakly equivalent for all b.

4.4 Lecture 15

4.4.1 Proof that symmetric products are a homology theory

Recall from the previous lecture that it remains to prove that

$$SP(A) \to SP(X) \to SP(X/A)$$

is a quasi-fibration. (Since it will then follow that $\cdots \to \pi_i(SP(A)) \ \to \cdots$) is a long exact sequence, which is the remaining axiom of homology theories that needed proving.)

We already have $SP(A) = p^{-1}(*)$.

Proof. Assume $A \hookrightarrow X$ is a cofibration. Define

$$\begin{aligned} B_n &= SP_n(X/A) \\ E_n &= p^{-1}(B_n). \end{aligned}$$

Then it is sufficient to show that $E_n \to B_n$ is a quasi-fibration for each n. We proceed by induction.

The base case, $n = 0$, is bookwork.

For the inductive step, write $B_n = V \cup (B_n \setminus B_{n-1})$ for some neighbourhood V of B_{n-1}. We wish to show that the restrictions of the inclusion map to V, $B_n \setminus B_{n-1}$, and $V \cap (B_n \setminus B_{n-1})$ are quasi-fibrations.

First we deal with V. Let $U = \{x_i : x_i \in N \text{ for some } i \leq n\}$, where $A \subset N \subset X$, where N is a neighbourhood of A admitting a retraction $f_t : X \to X$ sending N into A. Then $p(U) \subset B$ is a neighbourhood of B_{n-1} as desired.

To observe that restricting to V induces a quasi-fibration, we use the third property of the last lemma from the previous lecture. Using f_t, there exists a map $F_t : E \to E$ retracting U to E_{n-1}. This projects to a map $B_n \to B_n$ retracting V to B_{n-1}. As per the lemma, it remains to show that $F_1 : p^{-1}(b) \to p^{-1}(F_1(b))$ is a weak equivalence for all $b \in B_n$.

For any $w \in E_n$, we can write $w = \widehat{w}v$, where $\widehat{w} \in SP(X \setminus A)$ and $v \in SP(A)$. Then $F_1(w) = \widehat{F_1(\widehat{w})}v'v$. (This is slightly informal, but is suggestive and can be formalised.) Then

$$p^{-1}(b) = \widehat{w}SP(A), \quad p^{-1}(F_1(b)) = \widehat{F_1(\widehat{w})}SP(A).$$

Since v' depends only on b, one can use a path $v' \to *$ to make the map $\widehat{w}SP(A) \to \widehat{F_1(\widehat{w})}SP(A), v \mapsto v'v$ homotopic to the "identity". (This proof is difficult to write formally, but the result is "obvious" when thought about intuitively.) This completes the first part of the proof of the inductive step.

Next we require $E_n \setminus E_{n-1} \to B_n \setminus B_{n-1}$ to be a quasi-fibration. Observe that $B_n \setminus B_{n-1} = SP_n(X \setminus A)$. The above map is defined by $w \mapsto \widehat{w}$. There is an inclusion of the right side back into the left. Consider the induced map

$$\pi_i(E_n \setminus E_{n-1}, p^{-1}(b)) \to \pi_i(B_n \setminus B_{n-1}, b).$$

If these are all isomorphisms, $E_n \setminus E_{n-1} \to B_n \setminus B_{n-1}$ is a quasi-fibration as required. It is easily seen to be surjective. To see that it is injective, suppose $g : (D^n, \partial D^n) \to (E_n \setminus E_{n-1}, p^{-1}(b))$ lies in the kernel. But then null homotopy only changes coordinates in $X \setminus A$, and $E_n \setminus E_{n-1}$. $\square$

This completes the proof. In summary, we have proven the Dold-Thom theorem.

Chapter 5

Spectral sequences

Congratulations we've finished Hatcher yeet!

Or have we? It turns out that there's a secret chapter 5 of Hatcher, which covers much of the material we'll see in the upcoming lectures.

5.0.1 Motivation for spectral sequences

Suppose we have a chain of spaces

$$\cdots \subset X_{-i} \subset X_i \subset X_{i+1} \subset \cdots \subset X.$$

We call this a *filtration*. This is a common occurrence, for example expressing a topological space with its n-skeletons. If we know the homology of each X_i, can we understand the homology of X? This is our goal.

5.0.2 Basic terminology and notation for spectral sequences

Consider

$$\cdots \to H_{n+1}(X_p) \to H_{n+1}(X_p, X_{p-1}) \to H_n(X_{p-1}) \to H_n(X_{p-1}, X_{p-2}) \to \cdots .$$

This is unfortunately not exact in general. But what if we consider a grid?

$$
\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
{} \arrow[r, dotted] & H_{n+1}(X_p) \arrow[r, "j"] \arrow[d, dotted] & H_{n+1}(X_p, X_{p-1}) \arrow[r, "k"] \arrow[d, dotted] & H_n(X_{p-1}) \arrow[r, dotted] \arrow[d, "i"] & H_n(X_{p-1}, X_{p-2}) \arrow[r, dotted] \arrow[d, dotted] & {} \\
{} \arrow[r] & H_{n+1}(X_{p+1}) \arrow[r, dotted] \arrow[d] & H_{n+1}(X_{p+1}, X_p) \arrow[r, dotted] \arrow[d, dotted] & H_n(X_p) \arrow[r] \arrow[d, dotted] & H_n(X_p, X_{p-1}) \arrow[r] \arrow[d, dotted] & {} \\
{} \arrow[r, dotted] & H_{n+1}(X_{p+2}) \arrow[r] \arrow[d, dotted] & H_{n+1}(X_{p+2}, X_{p+1}) \arrow[r] \arrow[d, dotted] & H_n(X_{p+1}) \arrow[r, dotted] \arrow[d] & H_n(X_{p+1}, X_p) \arrow[r, dotted] \arrow[d, dotted] & {} \\
& {} & {} & {} & {} &
\end{tikzcd}
$$

We observe that some actual long exact sequences are hiding inside the grid! As shown with the solid arrows. We consistently use the notation i, j, k for maps as given. The long exact sequences look like "stairs". Define

$$A = \bigoplus_{n,p} H_n(X_p), \quad E = \bigoplus_{n,p} H_n(X_p, X_{p-1}).$$

Then our huge grid can be summarised in the following cute exact triangle!

$$
\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}
$$

Each map has a "bidegree", by inspecting how it changes the coordinates (n, p). Specifically, the bidegrees are:

$$\deg i = (0, 1), \quad \deg j = (0, 0), \quad \deg k = (-1, -1).$$

Keeping these bidegrees and the exact triangle in mind, a lot of theory can be developed without worrying about the notational struggles of the above grid.

Remark. We observe that $ik = ji = kj = 0$. However, there is one more composition we can make, namely $jk : E \to E$. We denote this by $d \coloneqq jk$. We observe that $d^2 = 0$, and in the setting of CW-complexes, this corresponds to the cell boundary map!

5.1 Lecture16

5.1.1 Basically homological algebra

Recall that at the end of the previous lecture, we defined the map

$$d = jk : E \to E$$

and observed that it was genuinely a differential in the sense that $d^2 = 0$. Using this we define a *derived exact triangle*:

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Here $E' = \ker d / \operatorname{im} d$, $A' = \operatorname{im} i = i(A)$, and $j' : A' \to E'$ is defined by

$$j'(a') = j'(ia) := j(a).$$

This is well defined by a standard homological algebra type argument, in which we show that if $ia_1 = ia_2$, then $j(a_1 - a_2) \in \operatorname{im} d$. Next we define $i' : A' \to A'$ to simply be the restriction of i, nice! Finally we define $k' : E' \to A'$ to be

$$k'([e]) = k(e) \in \ker j = \operatorname{im} i = A'.$$

Similarly if $[e] = 0$ then we can show that $e \in \ker k$.

Lemma 5.1.1. The derived triangle of an exact triangle is exact.

Proof. Firstly, $\ker j' = \operatorname{im} i'$. For one inclusion, suppose $a' \in A'$ is in the image of i'. Then $a' = iia$ for some a. But then

$$j'(a') = j'(iia) = j(ia) = 0.$$

For the opposite inclusion, suppose $j'a' = 0$. Then $ja \in \operatorname{im} d, \ldots$. Using standard chasing arguments the inclusion follows. Similarly exactness holds in the two other corners. $\square$

What really matters is keeping track of bidegrees. Do the maps in the derived exact triangle have the same bidegrees? We find that

$$\deg i' = (0, 1), \quad \deg j' = (0, -1), \quad \deg k' = (-1, -1).$$

But now we can continue inductively constructing derived triangles:

$$\begin{tikzcd} A^r \arrow[rr, "i^r"] & & A^r \arrow[dl, "j^r"] \\ & E^r \arrow[ul, "k^r"] & \end{tikzcd}$$

where $d^r : E^r \to E^r$ is defined to be $j^{r-1}k^{r-1}$, and the rest of the maps and spaces are defined in the same way as above.

Explicitly, unpacking the triangles, we have the following two yuge diagrams, with double arrows denoting the exact sequence traced out by the exact triangles:

$$
\begin{tikzcd}
& {} \arrow[d, Rightarrow] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
{} \arrow[r, dotted] & A_{n+1,p} \arrow[r, Rightarrow, "j"] \arrow[d, dotted] & E_{n+1,p} \arrow[r, Rightarrow, "k"] \arrow[d, dotted] & A_{n,p-1} \arrow[r, dotted] \arrow[d, Rightarrow, "i"] & E_{n,p-1} \arrow[r, dotted] \arrow[d, dotted] & {} \\
{} \arrow[r, Rightarrow] & A_{n+1,p+1} \arrow[r, dotted] \arrow[d, Rightarrow] & E_{n+1,p+1} \arrow[r, dotted] \arrow[d, dotted] & A_{n,p} \arrow[r, Rightarrow] \arrow[d, dotted] & E_{n,p} \arrow[r, Rightarrow] \arrow[d, dotted] & {} \\
{} \arrow[r, dotted] & A_{n+1,p+2} \arrow[r, Rightarrow] \arrow[d, dotted] & E_{n+1,p+2} \arrow[r, Rightarrow] \arrow[d, dotted] & A_{n,p+1} \arrow[r, dotted] \arrow[d, Rightarrow] & E_{n,p+1} \arrow[r, dotted] \arrow[d, dotted] & {} \\
& {} & {} & {} & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
{} \arrow[r, Rightarrow] & A^2_{n+1,p} \arrow[ur, dotted] \arrow[d, Rightarrow] & E^2_{n+1,p} \arrow[r, Rightarrow, "k'"] \arrow[d, dotted] & A^2_{n,p-1} \arrow[ur, dotted] \arrow[d, Rightarrow, "i'"] & E^2_{n,p-1} \arrow[r, Rightarrow] \arrow[d, dotted] & {} \\
{} \arrow[r, dotted] & A^2_{n+1,p+1} \arrow[r, dotted] \arrow[ur, Rightarrow, "j'"] \arrow[d, dotted] & E^2_{n+1,p+1} \arrow[r, dotted] \arrow[d, dotted] & A^2_{n,p} \arrow[ur, Rightarrow] \arrow[d, dotted] & E^2_{n,p} \arrow[r, Rightarrow] \arrow[d, dotted] & {} \\
{} \arrow[r, Rightarrow] & A^2_{n+1,p+2} \arrow[ur, dotted] \arrow[d, Rightarrow] & E^2_{n+1,p+2} \arrow[r, Rightarrow] \arrow[d, dotted] & A^2_{n,p+1} \arrow[ur, dotted] \arrow[d, Rightarrow] & E^2_{n,p+1} \arrow[r, Rightarrow] \arrow[d, dotted] & {} \\
& {} \arrow[ur, Rightarrow] & {} & {} \arrow[ur, Rightarrow] & {} &
\end{tikzcd}
$$

In the first diagram above, d maps in the horizontal direction, i.e. it has bidegree $(-1,-1)$. In the second diagram, d^2 maps diagonally upwards! It has bidegree $(-1,-2)$. In general d^r has bidegree $(-1,-r)$. We make some assumptions now to make spectral sequences manageable:

(i) Almost all maps i are isomorphisms. I.e. almost all $E'_{n,p} = E^2_{n,p}$ are trivial. Observe that it then makes sense to write $A_{n,\infty}, A_{n,-\infty}, E^{\infty}_{n,p}$.

(ii) $A_{n,\infty} = 0$

(iii) $A_{n,-\infty} = 0$.

Proposition 5.1.2. If (i) and (ii) above hold, then

$$E^\infty_{n,p} = F^p_n / F^{p-1}_n,$$

where $F^p_n = \mathrm{im}(A_{n,p} \to A_{n,\infty})$ is the map in the filtration

$$F^p_n \to F^{p+1}_n \to \cdots \to A_{n,\infty}.$$

If (i) and (iii) above hold, then

$$E^\infty_{n,p} = F^{n-1}_p / F^n_p,$$

where $F^{n-1}_p = \ker(A_{n,-\infty} \to A_{n-1,p})$.

Proof. Since these two statements are dual, we prove only one of them. Assume (i) and (ii) hold. Let r be large. We have a long exact sequence

$$\begin{tikzcd}
 & 0 = E^r_{n+1,p+r-1} \arrow[r, "k^r"] & A^r_{n,p+r-2} \arrow[dll, "i^r"] \\
A_{n,p+r-1} \arrow[r, "j^r"'] & E^r_{n,p} \arrow[r, "k^r"] & A^r_{n-1,p-1} \arrow[dll, "i^r"] \\
A^r_{n-1,p} \arrow[r, "j^r"'] & E^r_{n-1,p+1-r} = 0 &
\end{tikzcd}$$

where the start and end E^rs vanish by choosing large enough r. We then conclude that $A^r_{n-1,p}$ and $A^r_{n-1,p-1}$ vanish, so

$$E^r_{n,p} = \frac{A^r_{n,p+r-1}}{A^r_{n,p+r-2}}.$$

Since E stabilises we conclude that

$$E^\infty_{n,p} = \frac{F^p_n}{F^{p-1}_n}$$

as required. $\square$

5.2 Lecture 17

5.2.1 Quick review of spectral sequences to here

A quick review of things to here: the basic objects we want to study are filtrations of spaces, $\cdots X_p \subset X_{p+1} \subset \cdots$. We define $A_{n,p}$ to be $H_n(X_p)$, and $E_{n,p} = H_n(X_p, X_{p-1})$. We assume only finitely many of these are non-zero.

We then have an exact triangle

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

and define a differential by $d = kj$. From here we can construct *derived triangles*

$$\begin{tikzcd}
A^r \arrow[rr, "i"] & & A^r \arrow[dl, "j"] \\
& E^r \arrow[ul, "k"] &
\end{tikzcd}$$

where the maps have bidegree

$$\deg i = (0,1), \deg j = (0, -r+1), \deg k = (-1,-1), \deg d = (-1, -r),$$

and $E^{r+1}_{n,p} = \ker d^r / \operatorname{im} d^r$.

If $A_{n,-\infty} = 0$, then $E^\infty_{n,p} = F^p_n / F^{p-1}_n$, where $F^p_n = \operatorname{im}(A^1_{n,p} \to A^1_{n,p})$, and the F^p_n form a filtration. We write that $E^r_{n,p} \Rightarrow A_{n,\infty}$, and say it converges to $A_{n,\infty}$.

5.2.2 Arranging spectral sequences on a grid!

We originally had $E_{n,p}$, but now we substitute $n = p+q$, so that

$$E_{p+q,p} \rightsquigarrow E_{p,q}$$

is the new indexing. Then with these indices, $d^r : E^r_{p,q} \to E^r_{p,q}$ has bidegree $(-r, r-1)$. On page 1, we have

$$\begin{tikzcd}
E^1_{1,2} & E^1_{2,2} \arrow[l, "d^1"'] & E^1_{3,2} \arrow[l, "d^1"'] \\
E^1_{1,1} & E^1_{2,1} \arrow[l, "d^1"'] & E^1_{3,2}. \arrow[l, "d^1"']
\end{tikzcd}$$

On page 2, we have

$$\begin{tikzcd}
{} & E^2_{1,2} & E^2_{2,2} & E^2_{3,2} & {} \\
{} & E^2_{1,1} & E^2_{2,1} \arrow[ull, "d^2"] & E^2_{3,2}. \arrow[ull, "d^2"] & {} \arrow[ull, "d^2"]
\end{tikzcd}$$

Zooming all the way up to the infinitieth page, we have

$$\begin{matrix} E^\infty_{1,2} & E^\infty_{2,2} & E^\infty_{3,2} \\ & & \\ E^\infty_{1,1} & E^\infty_{2,1} & E^\infty_{3,2}. \end{matrix}$$

There are no non-zero differentials anymore, and the diagonals $E^\infty_{a,b}$ with $a+b=p+q$ form the building blocks of $A_{p+q,\infty}$.

5.2.3 Serre spectral sequence

Let $\pi : X \to B$ be a fibration, with B a path-connected CW complex. Let B^p denote the p-skeleton of B. Then (B, B^p) is p-connected. Set $X_p = \pi^{-1}(B^p) \subset X$. Then (X, X_p) is also p-connected!

We consider the spectral sequence $E_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$ built from the filtration $X_p \subset X_{p+1} \subset \cdots$. Observe that $E_{p,q} = 0$ whenever either p or q is negative, so this spectral sequence lives in the first quadrant.

Proposition 5.2.1. The E^2-page of the Serre spectral sequence is given by $E^2_{p,q} \cong H_p(B; H_q(F; G))$ if the $\pi_1(B)$ action on $H_k(F, G)$ is trivial. Here F denotes the fibre of $X \to B$.

Example. Suppose $X = B \times F$ so that the fibration is trivial. By the Künneth formula, $H_n(X; G) \cong \bigoplus_{p+q=n} H_p(B; H_q(F; G))$. But this is literally $H_n(X; G) \cong \bigoplus_{p+q=n} E^2_{p,q}$! Nice. This shows that the Serre spectral sequence converges on page two (differentials above page two are all trivial) so we can compute the homologies of X by just computing $E^2_{p,q}$.

Another example is $\mathbb{S}^1 \simeq K(\mathbb{Z}, 1) = \Omega K(\mathbb{Z}, 2) \to E_0(K(\mathbb{Z}, 2)) \to K(\mathbb{Z}, 2)$. (Recall that $E_0(K(\mathbb{Z}, 2))$ is the path space of $K(\mathbb{Z}, 2)$.) The path space is homotopic to a point. Thus we know the homology of the fibre ($\mathbb{S}^1$) and the homology of the total space ($*$), so hopefully we can use the Serre spectral sequence to determine the homologies of $K(\mathbb{Z}, 2)$. On the E^2 page, we have $E^2_{p,q} = H_p(K(\mathbb{Z}, 2); H_q(\mathbb{S}^1; G))$. We can now calculate the E^2 page: $E^2_{p,q}$ must vanish when $q < 0$ or $q > 1$. We can also observe that $E^2_{0,0} = E^2_{0,1} = \mathbb{Z}$. From this is follows that $E^2_{p,q} = \mathbb{Z}$ whenever p is even and $q \in \{0, 1\}$. Otherwise $E^2_{p,q}$ vanishes. Thus $H_p(K(\mathbb{Z}, 2); \mathbb{Z})$ is $\mathbb{Z}$ for p even, and 0 otherwise. This is $\mathbb{CP}^\infty$!

How about $\Omega\mathbb{S}^n \to E_0\mathbb{S}^n \to \mathbb{S}^n$? Again study the E^2 page. We have $E^2_{0,0} = \mathbb{Z}$, then a bunch of zeroes, then $E^2_{n,0} = \mathbb{Z}$. To kill $E^2_{n,0}$ on the next page, it must have an injective map out of it, which is necessarily d^n. Thus by computing,

$$H_g(\Omega\mathbb{S}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & g = kn - k \\ 0 & \text{otherwise.} \end{cases}$$

5.3 Lecture 18

5.3.1 Serre spectral sequence proof outline

Another example is as follows: consider

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

This gives rise to a fibration

$$K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}/4, 1) \to K(\mathbb{Z}/2, 1) \cong \mathbb{RP}^\infty.$$

Again look at the second page of the Serre spectral sequence. We find that

$$E^2_{p,q} = \begin{cases} \mathbb{Z} & p = q = 0 \\ \mathbb{Z}/2 & q = 0, p \text{ odd} \\ \mathbb{Z}/2 & q = 0, p \text{ even, non-zero} \\ \mathbb{Z}/2 & q \text{ odd} \\ 0 & \text{ otherwise.} \end{cases}$$

From here we argue that $H_n(K(\mathbb{Z}/4, 1)) = \mathbb{Z}/4$ if n is odd, and zero otherwise.

Recall that in the previous lecture we never actually proved the Serre spectral sequence. We do this now.

Proposition 5.3.1. The E^2-page of the Serre spectral sequence is given by $E^2_{p,q} \cong H_p(B; H_q(F; G))$ if the $\pi_1(B)$ action on $H_k(F, G)$ is trivial. Here F denotes the fibre of $X \to B$.

Proof. We know the E^1 page consists of rows as follows: We prove the vertical maps below are isomorphisms.

$$\begin{tikzcd} H_{p+q-1}(X_{p-1}, X_{p-2}) \arrow[d] & H_{p+q}(X_p, X_{p-1}) \arrow[l] \arrow[d] \\ H_{p-1}(B_{p-1}, B_{p-2}) \otimes H_q(F) \cong C_{p-1}(B) & H_p(B_p, B_{p-1}) \otimes H_q(F) \cong C_p(B) \arrow[l] \end{tikzcd}$$

We construct vertical isomorphisms as follows: consider all p-cells of B: lift $(D^p_\alpha, S^{p-1}_\alpha) \to (B_p, B_{p-1})$ to $(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha) \to (X_p, X_{p-1})$. We then obtain a map

$$\bigoplus_\alpha H_{p+q}(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha; G) \to H_{p+q}(X_p, X_{p-1}).$$

The claim is that this map is an isomorphism.

This is shown by considering tubular neighbourhoods and "thickening up spheres". By the excision theorem we can now conclude that

$$H_*(X_p, X_{p-1}; G) = H_*(\coprod_\alpha (\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha)).$$

Observe that there is an isomorphism

$$\bigoplus_\alpha H_q(F,G) \cong H_p(B_p, B_{p-1}) \otimes H_q(F,G).$$

We wish to show that there is an isomorphism

$$\bigoplus_\alpha H_{p+q}(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha; G) \cong \bigoplus_\alpha H_q(F,G).$$

We construct the isomorphism by writing $\partial D^p_\alpha = D^{p-1}_+ \cup_{\mathbb{S}^{p-2}} D^{p-1}_-$. This gives an isomorphism $H_{p+q}(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha; G) \cong H_{p+q-1}(\widetilde{S}^{p-1}_\alpha, \widetilde{D}^{p-1}_-; G)$. The right side is isomorphic to $H_{p+q-1}(\widetilde{D}^{p-1}_\alpha, \widetilde{S}^{p-2}_\alpha; G)$ by the excision theorem. Inductively we can continue this process until we have

$$H_{p+q}(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha; G) \cong H_q(\widetilde{D}^0; G) \cong H_q(F; G).$$

The last isomorphism holds when the $\pi_1(B)$ action is trivial, and hence this proves the proposition. □

5.3.2 Serre spectral sequence II: cohomological edition

Theorem 5.3.2. *For a fibration $F \to E \to B$ with $\pi_1(B)$ acting on $H_*(E)$ trivially, there is a spectral sequence with $H^{p,q}_1 = H^{p+q}(X_p, X_{p-1}, G)$, and $E^{p,q}_2 = H^p(B, H^q(F,G))$. We have $d^r : E^{p,q}_r \to E^{p+r,q-r+1}_r$, and $E^{p,n-p}_\infty \cong F^n_p / F^n_{p+1}$, where F is a filtration $0 \subset F^n_n \subset \cdots \subset F^n_0 = H^n(X,G)$.*

A nice property of the cohomological theory is that there is a multiplicative structure:

Theorem 5.3.3. *There is a multiplication map*

$$E^{p,q}_r \times E^{s,t}_r \to E^{p+s,q+t}_r$$

such that

(a) $d(xy) = (dx)y + (-1)^{p+q} x(dy)$

(b) On the second page, $E^{p,q}_2 \times E^{s,t}_2 \to E^{p+s,q+t}_2$ is $(-1)^{st}$ times the usual cup product:

$$H^p(B, H^q(F)) \times H^s(B, H^t(F)) \to H^{p+s}(B, H^{q+t}(F)).$$

(c) The product respects filtrations.

Observe that (a) and (b) determines the product everywhere: assume the product is defined on E_r. Choose $x, y \in E^{p,q}_{r+1}$, and let $x = [x'], y = [y'], xy = [x'y']$. Then if $x' - x'' = d\widetilde{x}$ and $y' - y'' = d\widetilde{y}$, one can show that $x'y' - x''y''$ lives in the image of d as required.

5.4 Lecture 19

5.4.1 More spectral examples

Example. Consider $K(\mathbb{Z},1) \cong \mathbb{S}^1 \to E \cong * \to K(\mathbb{Z},2) \cong \mathbb{CP}^\infty$. Inspect the second page of the Serre spectral sequence (of cohomology). Then we have $E_2^{0,0} = \mathbb{Z}, E_2^{0,1} = \mathbb{Z} \ni a$, where $x \mapsto xa$ is the map from $E_2^{0,0}$ to $E_2^{0,1}$. The differential from $E_2^{0,1}$ to $E_2^{2,0}$ maps a to some x_2. Continuing this process, $d(x_2 a) = x_4 = x_2^2$, and $x_{2n} = x_2^n$ in general. Therefore $H^*(K(\mathbb{Z},2),\mathbb{Z}) = \mathbb{Z}[x_2]$, where x_2 has degree 2.

Example. Next we investigate $H^*(\Omega\mathbb{S}^n,\mathbb{Z})$. Consider $\Omega\mathbb{S}^n \to E \cong * \to \mathbb{S}^n$. A similar derivation gives

$$E_n^{0,0} = \mathbb{Z}, E_n^{0,n-1} = \mathbb{Z} \ni a_1, E_n^{0,2n-2} = \mathbb{Z} \ni a_2$$

and so on. The differentials map from $E_2^{p,q} \to E_2^{p+n,q-n-1}$, sending a_1 to x, a_2 to $a_1 x$, and a_n to $a_{n-1}x$ more generally.

In the case where n is odd, we can derive that $a_1^k = k! a_k$. This gives $H^*(\Omega\mathbb{S}^1,\mathbb{Z}) = \Gamma_{\mathbb{Z}}[a]$, with the degree of $a = n-1$. "Divided polynomial something algebra".

In the case where n is even, the computations are a little more involved, but we can derive that $a_2^k = k! a_{2k}$. Then $H^*(\Omega\mathbb{S}^n,\mathbb{Z}) = \Lambda(a) \otimes \Gamma_{\mathbb{Z}}[b]$, where a has degree $n-1$, and b has degree $2n-2$.

Example. Homotopy of $\mathbb{S}^3$! Consider the map $\mathbb{S}^3 \to K(\mathbb{Z},3)$ which is an isomorphism on π_3. The map has a fibre X which is necessarily 3-connected. But $X \to \mathbb{S}^3$ is itself a fibration, with fibre $K(\mathbb{Z},2)$. We investigate this fibration.

A priori we know that $E_2^{0,0} = \mathbb{Z}$, and $E_2^{3,0} = \mathbb{Z}$. Since $K(\mathbb{Z},2)$ is $\mathbb{CP}^\infty$, we also have $\mathbb{Z} = E_2^{0,q} = E_2^{3,q}$ for all even q. Since X is 3-connected, $d : E_2^{0,2} \to E_2^{3,0}$ must be an isomorphism. The other differentials $d : E_2^{0,q} \to E_2^{3,q}$ are not necessarily isomorphisms, but we now see that they are multiplication by q. It follows that $H^{2n}(X,\mathbb{Z}) = 0$, while $H^{2n+1}(X,\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. Crazy!

By the universal coefficient theorem (since the cohomology is sufficiently sparse) we have isomorphisms between cohomology and homology. That is, $H_{2n-1}(X,\mathbb{Z}) = 0$, and $H_{2n}(X,\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. This shows that the first p-torsion in homology of X appears in degree $2p$. By "p-torsion Hurewicz", the first p-torsion in homology appears in the first p-torsion of homotopy. That is, $\pi_{2p}(X)$ is the first homotopy group with p-torsion! But this is the same as $\pi_{2p}(\mathbb{S}^3) = \mathbb{Z}/p\mathbb{Z}$. This also gives a stable homotopy group, $\pi_1^s(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$.

The above example made use of p-torsion Hurewicz. What is this?

Let $\mathcal{C} \subset \mathbf{Ab}$ be a full subcategory, i.e.

1. If $0 \to A \to B \to C \to 0$ and $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

2. If A, B are in $\mathcal{C}$, then their tensor products and Tor are in $\mathcal{C}$.

Theorem 5.4.1. *Let X be path connected, abelian. Suppose $\pi_i(X) \in \mathcal{C}$ whenever $i < n$. Then $\pi_i(X) \to H_i(X)$ is an isomorphism, modulo $\mathcal{C}$.*

Example. Let $\mathcal{C}$ denote the abelian groups without p-torsion.

Proof. Makes use of Postnikov towers and spectral sequences! $\square$

5.5 Lecture 20

Example. What is the cohomology $H^*(K(\mathbb{Z},3),\mathbb{Z})$? Consider the exact sequence

$$\mathbb{CP}^\infty \cong K(\mathbb{Z},2) \to E \cong * \to K(\mathbb{Z},3).$$

We investigate the second page of the Serre spectral sequence as usual. Then $E_2^{0,2n} \cong \mathbb{Z}$, with generators a^n. We find that $E_2 = E_3$ since the spectral sequence is concentrated on even rows, and on the third page, the differentials map from $E_3^{p,q} \to E_3^{p+3,q-2}$. By the Leibniz rule, we find that the differentials are multiplication maps: $d(a^k) = ka^{k-1}da = ka^{k-1}x$. (But the column of $p = 3$ is given by $E_3^{3,2n} \cong \mathbb{Z}$, with generators $a^n x$.)

Chapter 6

Localisation of spaces

Recall from algebra that localisations are the objects given by a universal construction to introduce units. For example,

$$\mathbb{Z}_{(p)} \subset \mathbb{Q}$$

consists of the integers where elements relatively prime to p are forced to be units, i.e. $\mathbb{Z}_{(p)}$ consists of fractions with denominators relatively prime to p. Localisations are local rings, in the sense that they have a unique maximal ideal.

Goal: can we invent an analogous notion for topological spaces? We wish to construct $X_{(p)}$ such that

$$\pi_*(X) \otimes \mathbb{Z}_{(p)} \cong \pi_*(X_{(p)}), \quad H_*(X) \otimes \mathbb{Z}_{(p)} \cong H_*(X_{(p)}).$$

Remark. Localisation of rings is an exact functor.

Definition 6.0.1. Let X be an abelian space. X is said to be $\mathcal{P}$-local if $\pi_i(X)$ is a $\mathbb{Z}_{\mathcal{P}}$-module. $X \to X'$ is a $\mathcal{P}$-localisation if $\pi_*(X)\otimes\mathbb{Z}_{\mathcal{P}}$ maps isomorphically onto $\pi_*(X')\otimes\mathbb{Z}_{\mathcal{P}} \cong \pi_*(X')$.

In the above, $\mathcal{P}$ denotes any set of prime numbers in $\mathbb{Z}$. $\mathbb{Z}_{\mathcal{P}}$ is the collection of fractions with denominators relatively prime to all of $\mathcal{P}$.

Theorem 6.0.2. *(a) For every abelian space, there is a $\mathcal{P}$-localisation $X \to X'$.*

(b) $X \to X'$ is a $\mathcal{P}$-localisation if and only if $H_*(X')$ is a $\mathcal{P}$-module and $H_*(X) \otimes \mathbb{Z}_{\mathcal{P}} \to H_*(X')$ is an isomorphism.

(c) This gives a homotopy functor from abelian topological spaces to $\mathcal{P}$-local topological spaces. In particular, X' is well defined up to homotopy.

Corollary 6.0.3. *As a corollary of (b), $(\mathbb{S}^n)_{(p)} \cong M(\mathbb{Z}_{(p)}, n)$.*

Remark. Let $F \to E \to B$ be a fibre sequence. If two out of three are $\mathcal{P}$-local, then so is the third.

Proof. We begin by proving (a) assuming the forwards direction of (b). We use Postnikov towers of X. Consider the following diagram:

$$\begin{tikzcd}
 & X_3 \arrow[d] & {} \arrow[d, dotted, "5"] & \\
K(\pi_2, 4) & X_2 \arrow[l] \arrow[d] \arrow[r, dotted, "3"] & X_2' \arrow[d, dotted, "3"] \arrow[r, dotted, "4"] & {} \\
K(\pi_1, 3) & X_1 = K(\pi_1(X), 1) \arrow[l] \arrow[r, dotted, "1"] & X_1' = K(\pi_1(X)_{(p)}, 1) \arrow[r, dotted, "2"] & K(\pi_{2\mathcal{P}}, 3)
\end{tikzcd}$$

Consider the natural map $\pi_1(X) \to \pi_1(X)_{\mathcal{P}}$. Then $X_1 \to X_1'$ a $\mathcal{P}$-localisation implies that $H^*(X', X; A) = 0$, where A is any $\mathbb{Z}_{\mathcal{P}}$-module. We have a map 1 as above.

Next we construct 2 by using the natural map $\pi_2 \to \pi_{2\mathcal{P}}$ and using results from obstruction theory. Now the maps 3 arise by constructing X_2' as the fibre, and continue inductively for the rest of the maps to build a Postnikov tower of X'. This defines X'.

$\sim$

Next we prove the forwards direction of (b). One can verify that it is indeed true for $\mathbb{S}' = K(\mathbb{Z}, 1)$, and $X' = M(\mathbb{Z}\mathcal{P}, 1)$. Then it holds for $X = K(\mathbb{Z}_{\mathcal{P}}, 1)$ and $X = K(\mathbb{Z}_{(p)}, 1)$, with p not in $\mathcal{P}$. By the Künneth formula, the result is true for $K(\pi, 1)$ with π a finitely generated abelian group.

Fact: Suppose $F \to E \to B$ is a fibration and $\pi_1(B)$ acts trivially on $H_*(E)$. Then if two of $\widetilde{H}_*(F), \widetilde{H}_*(E), \widetilde{H}_*(B)$ are $\mathcal{P}$-local, then so is the third. This follows from the Serre spectral sequence.

Using this, since we have $K(\pi, n-1) \to E \cong * \to K(\pi, n)$, using Postnikov towers allows the result to inductively generalise to all X, since we also have $X_{n+1} \to X_n \to K(\pi, n+2)$.

$\sim$

To finish part (b), we must prove the reverse direction. Consider a localisation $X \to X''$ and a homology localisation $X \to X'$. Recall that $H^*(X', X, A) = 0$ for all $\mathbb{Z}_{\mathcal{P}}$-modules A. By obstruction theory this allows us to lift $X \to X'$ to the map $X \to X''$. The lift $X' \to X$ is a homotopy equivalence by p-torsion Hurewicz from the previous lecture.

Part (c) is left as an exercise. $\square$

Chapter 7

Model Categories

7.1 Lecture 21

7.1.1 Initial definitions

A problem concerning homotopies is the following. What are the pushouts of the following two diagrams?

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] \arrow[d] & * \arrow[d, dotted] \\ * \arrow[r, dotted] & ? \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^1 \arrow[r] \arrow[d] & D^2 \arrow[d, dotted] \\ D^2 \arrow[r, dotted] & ? \end{tikzcd}$$

It's clear that the two diagrams are homotopy equivalent. But what are their pushouts? The pushout of the first diagram is $*$, while the pushout of the second diagram is $\mathbb{S}^2$. Oh no! These aren't homotopy equivalent.

How do we fix this? We must understand homotopy structures on categories.

Definition 7.1.1. A *model category* is a category $\mathcal{C}$ such that

- $\mathcal{C}$ has all small limits and colimits
- There are classes of morphisms called *weak equivalences, fibrations*, and *cofibrations.* These necessarily satisfy the two-out-of-three property, namely if any two maps in a commuting triangle belong to one of the classes, so does the other.
- These classes are all closed under retracts.
- Trivial cofibrations have the left-lifting property with respect to fibrations, and cofibrations have the left-lifting property with respect to trivial fibrations.

- There is a functorial factorisation of any $f : X \to Y$ into a composition $X \hookrightarrow Y' \twoheadrightarrow Y$ where the former is a cofibration and the latter a weak equivalence, as well as a factorisation $X \hookrightarrow X' \twoheadrightarrow Y$ where the former is a weak equivalence and the latter a fibration.

Example.

- The category of topological spaces is a Model category.
- There are three trivial model category structures on any category.
- If a category is a model category, so is its opposite category.

Remark. We hereafter use $\twoheadrightarrow$ to denote fibrations, and $\hookrightarrow$ to denote cofibrations.

Every model category has initial and terminal objects, denoted $\varnothing$ and $*$ respectively. X is called *fibrant* if there is a fibration $X \twoheadrightarrow *$, and *cofibrant* if $\varnothing \hookrightarrow X$.

There are functors $Q, R : \mathcal{C} \to \mathcal{C}$ such that $\varnothing \hookrightarrow QX \twoheadrightarrow X$, where the fibration is a weak equivalence. On the other hand, R satisfies $X \hookrightarrow RX \twoheadrightarrow *$, where the former is a weak equivalence.

We can form "pointed" categories by modding out by the terminal object.

Lemma 7.1.2. Let $\mathcal{C}$ be a model category. Then f is a cofibration if and only if f has the left lifting property with respect to trivial fibrations. Similarly, f is a trivial cofibration if and only if it has the left lifting property with respect to fibrations.

Remark. This shows that the definition of a model category is overdetermined: if we know the weak equivalences and cofibrations, we know the fibrations etc.

Proof. One direction for each claim is immediate by the definition.

For the converse, assume $f : A \to B$ has the left lifting property with respect to trivial fibrations. By expressing what this means in terms of diagrams, the result follows $\square$

Corollary 7.1.3. *Cofibrations are preserved under pushout, and fibrations are preserved under pullback.*

Theorem 7.1.4 (Ken Brown's Lemma)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories. If F sends trivial cofibrations between cofibrant objects to weak equivalences, then F sends all weak equivalences between cofibrant objects to weak equivalences. The dual statement also holds.*

Proof. If $f : A \to B$ is a weak equivalence and A, B are cofibrant, consider the following diagram:

$$
\begin{tikzcd}
A \arrow[d, hook] \arrow[dr, hook, dotted, "\sim"] \arrow[drr, bend left, "\sim"] & & \\
A \sqcup B \arrow[r, hook] & B' \arrow[r, two heads, "\sim"] & B \\
B \arrow[u, hook] \arrow[ur, hook, dotted, "\sim"] \arrow[urr, bend right, "\sim"'] & &
\end{tikzcd}
$$

The dotted arrows are weak equivalence cofibrations by two-out-of-three. Apply the functor and apply two-out-of-three properties again to abtain the conclusion. □

7.1.2 Homotopy categories

Definition 7.1.5. If $\mathcal{C}$ is a model category, then its *homotopy category* $\operatorname{Ho}\mathcal{C}$ has objects $\operatorname{ob}(\mathcal{C})$, and morphisms $\operatorname{mor}(\operatorname{Ho}\mathcal{C})$ the morphisms in $\mathcal{C}$ along with declared inverses for weak equivalences.

Remark. It isn't clear that $\operatorname{Ho}\mathcal{C}$ is is a set! There are some set theoretic issues that need to be resolved.

Proposition 7.1.6. Let $\mathcal{C}_c, \mathcal{C}_f, \mathcal{C}_{cf}$ be the subcategories consisting of cofibrant, fibrant, and cofibrant-fibrant objects. Then $\operatorname{Ho}\mathcal{C}$ is isomorphic to $\operatorname{Ho}\mathcal{C}_f$ and $\operatorname{Ho}\mathcal{C}_c$, while $\operatorname{Ho}\mathcal{C}$ is equivalent to $\operatorname{Ho}\mathcal{C}_f$ and $\operatorname{Ho}\mathcal{C}_c$.

Proof. We give a proof strategy that $\operatorname{Ho}\mathcal{C}$ is a set. Define "cylinder" and "path" objects. This gives a notion of left and right homotopies between maps, and then homotopy equivalences. On fibrant-cofibrant objects, left and right homotopy homotopies can be shown to be the same, so that homotopy equivalence is the same as weak equivalence. It follows that the homotopy category is obtained as a quotient of the original category by homotopy equivalence, which is a set. □

7.2 Lecture 22

7.2.1 Quillen functors

Definition 7.2.1. Let $\mathcal{C}, \mathcal{D}$ be model categories. $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor if it is a left adjoint and preserves trivial cofibrations. Similarly $U : \mathcal{D} \to \mathcal{C}$ is right Quillen if it is right adjoint and preserves trivial fibrations.

Lemma 7.2.2. If $F : \mathcal{C} \leftrightarrow \mathcal{D} : U$ is an adjunction, then L is left Quillen if and only if U is right Quillen.

Proof. Exercise. Follows from diagram chasing. □

Remark. By Ken Brown's lemma, Quillen functors preserve weak equivalences between (co)fibrant objects.

Example. $c : \mathcal{C} \to \mathcal{C}^I$: lim is an adjunction of Quillen functors, where c is the constant functor and $\mathcal{C}^I$ is the diagram category.

7.2.2 Derived functors

Let $\mathcal{C}, \mathcal{D}$ be model categories. If $F : \mathcal{C} \to \mathcal{D}$ is left Quillen, then

$$\begin{tikzcd} \operatorname{Ho}\mathcal{C} \arrow[r, "Q"] \arrow[rr, bend right, "LF"'] & \operatorname{Ho}\mathcal{C}_c \arrow[r, "F"] & \operatorname{Ho}\mathcal{D} \end{tikzcd}$$

is the *Left derived functor.* We must jump through the cofibre replacement to ensure that the maps are well defined. Similarly, if $U : \mathcal{D} \to \mathcal{C}$ is right Quillen, then

$$\begin{tikzcd} \operatorname{Ho}\mathcal{D} \arrow[r, "R"] \arrow[rr, bend right, "RU"'] & \operatorname{Ho}\mathcal{D}_f \arrow[r, "U"] & \operatorname{Ho}\mathcal{C} \end{tikzcd}$$

is the *Right derived functor.*

Theorem 7.2.3. *Left and right derived functors are well defined and sufficiently natural.*

Lemma 7.2.4. If $F : \mathcal{C} \leftrightarrow D : U$ is a Quillen pair, there is an adjunction $LF : \operatorname{Ho}\mathcal{C} \leftrightarrow \operatorname{Ho}\mathcal{D} : RF$

Proof. We want to show that $\operatorname{Ho}\mathcal{D}(LFX, Y) \cong \operatorname{Ho}\mathcal{C}(X, RUY)$. The left side is a quotient of $\mathcal{D}(FQX, RY)$, and the right side is a quotient of $\mathcal{C}(QX, URY)$. Prior to quotients, there is an isomorphism $\mathcal{D}(FQX, RY) \to \mathcal{C}(QX, URY)$ since F, U are adjoint. It remains to verify that the equivalence relations are respected by this isomorphism.

To this end, suppose $\varnothing \hookrightarrow A \in \mathcal{C}$, and $B \twoheadrightarrow * \in \mathcal{D}$, and $f, g : FA \to B$ are homotopic. Then we can show that φf is homotopic to φg by considering the appropriate diagram and applying the functor U. Similarly by applying F, we find that two homotopic maps $A \to UB$ come from homotopic maps. □

7.2.3 Quillen equivalences

Definition 7.2.5. "Weak equivalences on the right come from weak equivalences on the left."

A Quillen adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : U$ is a Quillen equivalence if for all $\varnothing \hookrightarrow X \in \mathcal{C}$ and $Y \twoheadrightarrow * \in \mathcal{D}$, $FX \to Y$ is a weak equivalence if and only if $X \to UY$ is a weak equivalence.

Proposition 7.2.6. Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : U$ be a Quillen adjunction. The following are equivalent:

(a) (F, U) is a Quillen equivalence.

(b) $X \to UFX \to URFX$ and $FQUY \to FUY \to Y$ are equivalences for all $X \in \mathcal{C}_c$, $Y \in \mathcal{D}_f$.

(c) LF, RU are adjoint equivalences $\operatorname{Ho}\mathcal{C} \leftrightarrow \operatorname{Ho}\mathcal{D}$.

Proof. (a) to (b): suppose (F, U) is a Quillen equivalence. Note that $FX \to RFX$ is a weak equivalence, so the adjoint $X \to URFX$ is a weak equivalence. Similarly $FQUY \to FUY \to Y$ is a weak equivalence.

(b) to (a): Let $f : FX \to Y$ be a weak equivalence. We must show that $\varphi f : X \to UY$ is a weak equivalence. This follows by inspecting the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend left, "\varphi f"] \arrow[d, "="] & UFX \arrow[r] \arrow[d] & UY \arrow[d, "\sim"] \\
X \arrow[r, "\sim"] & URFX \arrow[r] & URY.
\end{tikzcd}$$

Finally the equivalence of (b) and (c) comes from the diagram

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[rrd, "\sim"] & QX \arrow[r, "\sim"] & URFX \arrow[d, "="] \\
 & & (RU)(LF)X
\end{tikzcd}$$

□

Corollary 7.2.7. *Suppose F, U are Quillen adjoints. The following are equivalent:*

(a) (F, U) is a Quillen equivalence.

(b) F reflects equivalences between cofibrant objects and $FQUY \to Y$.

(c) U reflects equivalences between fibrant objects analogously.

7.3 Lecture 23

7.3.1 Small object argument

An ordinal is a category λ with exactly one morphism between all objects:

$$\bullet \to \bullet \to \bullet \to \cdots .$$

These give λ-sequence in any $\mathcal{C}$

$$X_1 \to X_2 \to X_3 \to \cdots .$$

These sequences might be very large (they're rarely countable). We will pretend $\lambda = \infty$ (countable infinity) for simplicity.

Another heuristic definition: $A \in \mathcal{C}$ is *small* with respect to a class of morphisms $\mathcal{D}$ if there is λ such that for λ-or-bigger-sequences $X_0 \to X_1 \to \cdots$, any map $A \to \mathrm{colim}_\lambda X_\beta$ factors through $A \to X_\beta$.

Example. Every set is small.

Proof. Use $\lambda = |S|$. $\square$

Definition 7.3.1. If I is a class of morphisms in $\mathcal{C}$, I-inj is defined to be the collection of morphisms f with the following lifting property:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g\in I"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[ur, dotted, "\exists"] & \bullet \end{tikzcd}$$

Similarly I-proj is the collection of morphisms with the dual property.

I-cof is defined to be (I-inj)-proj, and I-fib is defined to be (I-proj)-inj.

Example. If $\mathcal{C}$ is a model category, and I are the trivial cofibrations, then I-inj are the fibrations and I-proj are the trivial cofibrations.

The idea is to use this to "generate" fibrations and cofibrations on a model category. Lifting properties will be easy to verify, but functoriality might be hard.

Definition 7.3.2. Let $\mathcal{C}$ be a category containing all small colimits, I a class of maps. Then I-cell is the class of "relative I-cell complexes", i.e. compositions of pushouts

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g\in I"'] & X_\beta \arrow[d] \\ C_{\beta+1} \arrow[r] & X_{\beta+1} \end{tikzcd}$$

Remark. These should intuitively be thought of as something like CW-complexes.

Lemma 7.3.3. *I-cell is a subset of I-cof.*

Proof. Follows from universal property of pushouts. ☐

Theorem 7.3.4 (Small object argument)**.** *Let $\mathcal{C}$ be a category with small colimits, I a class of maps. Suppose the domains of I are small relative to I-cell. Then there is a functorial factorisation (γ, δ) such that $f = \delta(f) \circ \gamma(f)$ and $\gamma(f) \in I$-cell, $\delta(f) \in I$-inj.*

Proof. Let $f : X \to Y$. We will define functorially

$$X = Z_0^f \to Z_1^f \to \cdots \to Z_\beta^f \to \cdots$$

with $Z_\beta^f \to Y$, and $Z_\beta^f \to Z_{\beta+1}^f \in I$-cell. Set $\gamma(f) : X \to \mathrm{colim}_\beta Z_\beta^f$, and $\delta(f) : \mathrm{colim}_\beta Z_\beta^f \to Y$. Why can we do this? Consider the collection of squares

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "g \in I"] & Z_\beta^f \arrow[d] \\ B_i \arrow[r] & Y \end{tikzcd}$$

and glue them together to form

$$\begin{tikzcd} \coprod A_i \arrow[r] \arrow[d, "\coprod g"] & Z_\beta^f \arrow[d] \arrow[ddr, bend left] & \\ \coprod B_i \arrow[r] \arrow[drr, bend right] & Z_{\beta+1}^f \arrow[dr, dotted] & \\ & & Y. \end{tikzcd}$$

Then it is immediate that $\gamma(f)$ is in I-cell. It remains to verify that $\delta(f)$ is in I-inj. Consider the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g \in I"] & \mathrm{colim}_\beta Z_\beta^f \arrow[d, "\delta(f)"] \\ B \arrow[r] & Y. \end{tikzcd}$$

By smallness, this factors as follows. Observe that the horizontal map at the bottom is exactly a desired "lift".

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g\in I"] & Z^f_\beta \arrow[d, "\delta(f)"] \arrow[dr, bend left] & \\
B \arrow[r, "\text{lift}"] \arrow[drr, bend right] & Z^f_{\beta+1} \arrow[dr] & \\
& & Y
\end{tikzcd}$$

$\square$

Definition 7.3.5. If $\mathcal{C}$ is a model category, it is said to be *cofibrantly generated* if there are classes of maps I, J such that

- the domains of I are small relative to I-cell,
- domains of J are small relative to J-cell,
- fibrations are J-inj,
- cofibrations are I-cof,
- trivial fibrations are I-inj.

Remark. All known model categories are cofibrantly generated.

Remark. There is no analogue for "fibrantly generated" model categories, as it requires the notion of co-small objects, which doesn't appear in nature.

Theorem 7.3.6. *Let $\mathcal{C}$ be a category with small (co)limits. Let $W \subset \mathcal{C}$ be a subcategory, I, J classes in $\mathcal{C}$. Then $\mathcal{C}$ is a cofibrantly generated model category if*

- *W satisfies two-out-three,*
- *Objects are small with respect to I, J,*
- *J-cell is a subset of $W \cap I$-cof,*
- *I-inj is a subset of $W \cap J$-inj,*
- *Either one of the above is an equality.*

Proof. Most parts of the proof follow from the small object argument applied to I and J. Left to verify that lifting properties are verified as below:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\in I\text{-cof}"'] & \bullet \arrow[d, "\in J\text{-inj}\cap W"] \\
\bullet \arrow[r] \arrow[ur, dotted] & \bullet
\end{tikzcd}$$

□

7.4 Lecture 24

7.4.1 Examples of model categories

Example. Let R be a ring. Then $Ch(R)$ is a model category, with $I = \{\mathbb{S}^{n-1} \to D^{n-1}\}$, where $\mathbb{S}^{n-1}$ is defined to be the complex $0 \to R \to 0$ for some degree $n-1$ R, and D^n is $0 \to R \to R \to 0$, where R is degree n.

Then $J = \{0 \to D^n\}$, and $W = \{f : H_*(f) \text{ are isomorphisms}\}$. We find that $X \to Y$ is a fibration if and only if $X_n \to Y_n$ is a surjection for each n, and if X is cofibrant, then all X_n are projective. Conversely, if X is left fibrant and each X_n is projective, then X is cofibrant.

$\iota : A \to B$ is cofibrant if and only if it splits dimensionwise with cofibrant cokernels.

Example. The canonical examples are of course created from Top. Top itself is a model category, with W the weak equivalences, $I = \{\mathbb{S}^n \to D^n\}$, $J = \{D^n \to D^n \times I\}$. Note that every topological space is fibrant.

Remark. Top is only small with respect to injections.

7.4.2 Simplicial sets

Remark. There are many different model category structures on simplicial sets. The two "important ones" are precisely the structures that turn simplicial sets into topological spaces, and one that gives rise to all infinity categories.

Definition 7.4.1. We write $[n] = [0 \to 1 \to 2 \to \cdots \to n]$. Then Δ denotes

$$\Delta = \{[n] \text{ with order preserving maps}\}.$$

We have maps between $[n]$ and $[n+1]$, the number of maps can be counted. e.g. there are two maps $[0] \to [1]$ called "degeneracy maps", and two maps $[2] \to [1]$ called "face maps".

Definition 7.4.2. The category of simplicial sets SSet is defined by

$$\mathrm{SSet} = \mathrm{Set}^{\Delta^{\mathrm{op}}}.$$

Remark. Each X_0 is interpreted as a point, X_1 as an interval, X_2 as a triangle, and so on.

Definition 7.4.3. More generally, given any category $\mathcal{C}$, it has a corresponding *simplicial category*,

$$\mathrm{S}\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}.$$

Lemma 7.4.4. Every simplicial set is small.

We have a map $\Delta \to \mathrm{SSet}$, and hence a map $\Delta^{\mathrm{op}} \times \Delta \to \mathrm{Set}$, which we can write by $([n],[m]) \mapsto \Delta([n],[m])$. In particular, we have $(\Delta[n])_n = \Delta([n],[n])$ which consists of one non-degenerate n-simplex, and $(\Delta[n]), n-1$ consists of $n+1$ non-degenerate $n-1$-simplies. This is exactly an n-simplex in the topological sense!

As an example,

$$\begin{aligned}\Delta[0] &\cong \cdot \\ \Delta[1] &\cong - \\ \Delta[2] &\cong \triangle\end{aligned}$$

$\partial\Delta[n]$ denotes the boundary of $\Delta[n]$, and $\Lambda^k[n]$ denotes the kth *horn* of $\Delta[n]$.

Δ is formally a functor. Given any simplicial set K,

$$\Delta K = \{\Delta[n] \to K, \text{ for all } n\}.$$

This is called the *triangulation* of K. The colimit of $(\Delta K \to \mathrm{SSet})$ is K.

Remark. This are looking pretty topological, but we still don't have a precise way of realising simplicial sets as topological spaces. Time to remedy this!

Definition 7.4.5. Let $|\Delta[n]| \subset \mathbb{R}^n$ be defined by

$$|\Delta[n]| = \{x_0 + \cdots + x_n = 1 : x_i \geq 0\}.$$

The *geometric realisation* map from Δ to Top is

$$|\Delta[-]| : \Delta \to \mathrm{Top}.$$

Given any simplicial set K, its geometric realisation is

$$|K| \coloneqq \mathrm{colim}_{\Delta K}| - |.$$

Proposition 7.4.6. Geometric realisation is a left adjoint to the Sing functor, where $\mathrm{Sing}(X) = \mathrm{Maps}(|\Delta[n]|, X)$.

Proof. The key idea: it suffices to verify the adjunction for smaller pieces of simplicial sets, since every simplicial set can be realised as a colimit. That is, it remains to show

$$\mathrm{SSet}(\Delta[n], \mathrm{Sing}(|\Delta[m]|)) = \mathrm{Top}(|\Delta[n]|, |\Delta[m]|).$$

This follows from unpacking some definitions $\square$

Lemma 7.4.7. Geometric realisation from SSet to compactly generated topological spaces preserves products.

Proof. Proof idea: it suffices to show that $|\Delta[n] \times \Delta[m]| = |\Delta[n]| \times |\Delta[m]|$. Combinatorics. $\square$

Remark. Geometric realisation preserves all small limits and colimits.

7.5 Lecture 25

This lecture and all subsequent lectures were cancelled due to the COVID-19 outbreak.