THE ISOMORPHISM FROM KNOT CONTACT HOMOLOGY TO STRING HOMOLOGY

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Abstract. We define a version of knot contact homology, and recall the definition of string homology. We construct a chain map from the knot contact setting to the string setting, and claim that it's an isomorphism on homology. I was going to have a nice example to carry through, but unfortunately it was harder to compute than I expected.

1. Knot contact homology: the idea

The general setup for Legendrian contact homology is as follows:

(1) Let L be a Legendrian submanifold of a contact manifold M. (This means the tangent spaces of L lie inside the contact distribution.) A *Reeb chord* is a geodesic segment of M that starts and ends on L. (More specifically, it's a flow of the Reeb vector field R; $R(\alpha) = 1, dR(\alpha, -) = 0$.)
(2) We can define a grading on the Reeb chords, specifically the Maslov index.
(3) The *Algebra of chords* is now a graded algebra. To create a homology theory we need a chain map. This is defined by counting the number of holomorphic curves which start at a given chordα , and end at a chain $\mathbf{b}$ of chords, where $|a| - |\mathbf{b}| = 1$. Now the chain map is

$$\partial(a) = \sum_{|a|-|\mathbf{b}|=1} \#\{\text{holomorphic curves from } a \text{ to } \mathbf{b}\}\mathbf{b}.$$

As an example, we can readily define Legendrian contact homology for *Legendrian knots*. That is, if a knot has tangent lines inside the standard contact distrubtion of $\mathbb{R}^3$, then we obtain a *Legendrian isotopy invariant* of the Legendrian knot using the above framework.

In our case, we want to define a homology theory for all smooth knots, not just Legendrian ones. What do we do?

Definition 1.1. Let $K \subset Q = \mathbb{R}^3$ be a smooth knot. We write U^*Q to denote the *unit cotangent bundle* of Q, that is, the $\mathbb{S}^2$-bundle over $\mathbb{R}^3$ consisting of unit covectors in the cotangent bundle. U^*Q is naturally a contact manifold: any point in $T*Q$ is of the form (q, p) where q is the point and $p \in T^*_q Q$. The *tautological form* is the one-form $\lambda = p_i dq^i$ on $T*Q$. For a coordinate free description, we can write $\lambda = p \circ d\pi$, where $\pi : T^*Q \to Q$ is the bundle projection. The restriction of λ to U^*Q is a contact form.

Let, we define Λ_K to be the unit conormal bundle of K, that is, the bundle of unit-length normal vectors to K. Then Λ_K is a submanifold of U^*Q.

Proposition 1.2. For any knot K, Λ_K is a Legendrian submanifold of U^*Q. (It follows that we can attmept to do Legendrian contact homology for any knot.)

Proof. Let $v \in T(\Lambda_K)$. Then $d\pi : T(U^*Q) \to TQ$ sends v to TK. However, by definition, v is conormal to K so it vanishes on TK. Thus $\lambda(v) = (p \circ d\pi)(v) = 0$. □

It turns out that to construct an isomorphism $H_0^{contact}(K) \to H_0^{string}(K)$, we can't use the oridinary notion of knot contact homology from above. Instead we need to use a refinement. The next section will detail how we define the refined version of knot contact homology.

2. Knot contact homology: details

Our setup is as follows: we have a knot K in $Q = \mathbb{R}^3$, and we upgrade to $\Lambda_K \subset U^*Q$. Λ_K is a Legendrian submanifold, where U^*Q has the contact form λ.

The *Reeb vector field* R is the unique vector field satisfying $\lambda(R) = 1, d\lambda(R, -) = 0$. A *Reeb chord* $a : [0, T] \to U^*Q$ is a solution to $a' = R, a(0), a(T) \in \Lambda_K$.

Proposition 2.1. Reeb chords of Λ_K are in bijective correspondence with binormal chords of K. (That is, geodesic segments of $\mathbb{R}^3$ meeting K orthogonally at endpoints.) Moreover, under this correspondence, every Reeb chord for a generically embedded knot K corresponds to a critical point of the Morse function $d : K \times K \to \mathbb{R}$. (That is, the distance function.)

Example. A circle in $\mathbb{R}^3$ is not generic. An ellipse is a generic embedding of the unknot, with two critical values of the distance function: the minor axis is a local minimum of the distance function and thus an index 0 critical point. The major axis is a local maximum and thus an index 2 critical point.

Proposition 2.2. Let a be a Reeb chord of Λ_K. Then the Maslov index of a satisfies

$$\mu(a) = \text{ind}(a) + 1$$

where $\text{ind}(a)$ is the Morse index of a. For defining knot contact homology, we'll actually use the Morse index rather than the Masloc index; the degree $|a| = \text{ind}(a)$.

Example. The elliptic unknot has two Reeb chords, with Maslov indices 1 and 3.

Definition 2.3. Let $\mathbf{a} = a_1 a_2 \cdots a_n$ be a word in the Reeb chords of Λ_K. The *degree* of $\mathbf{a}$ is the sum of the degrees of a_i. Given a chord a and word $\mathbf{b}$, we write $\mathcal{M}(a; \mathbf{b})$ to denote the moduli space of marked J-holomorphic disks

$$(D, \partial D) \to (\mathbb{R} \times U^*Q, \mathbb{R} \times \Lambda_K)$$

where a positive puncture maps asymptotically over a, and ℓ negative punctures map over $b_1, \ldots, b_\ell$.

We need to multiply the codomain by $\mathbb{R}$ to symplectify it, and then we can speak of J-holomorphic curves.

Proposition 2.4. The moduli space is $|a| - |\mathbf{b}|$ dimensional. Notice that the extra $\mathbb{R}$ factor in the codomain means there's a canonical $\mathbb{R}$ action on the moduli space. Now

$$\dim \mathcal{M}(a; \mathbf{b})/\mathbb{R} = |a| - |\mathbf{b}| - 1.$$

At this point we have the ingredients for classical knot contact homology: our algebra would be generated by Reeb chords, and the boundary map would count the size of the moduli spaces. We now start to make some further definitions to encode more information in our boundary map.

Definition 2.5. A *Reeb string* with ℓ chords is an expression

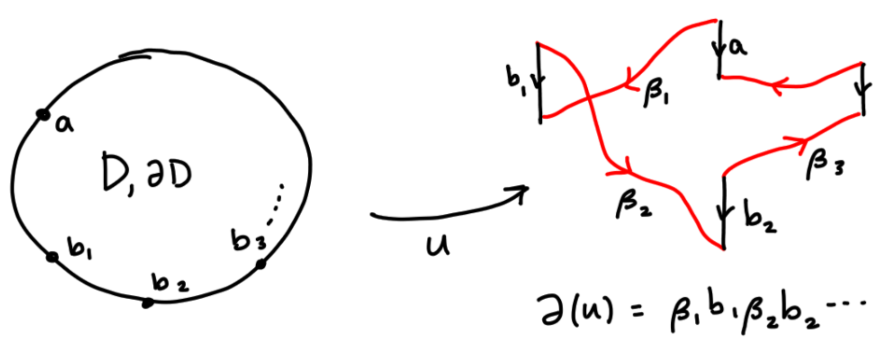
$$\alpha_1 a_1 \cdots \alpha_\ell a_\ell \alpha_{\ell+1},$$

where $a_1, \ldots, a_\ell$ are Reeb chords, $a_i : [0, T_i] \to U^*Q$, and α_i are paths in Λ_K from $a_{i-1}(0)$ to $a_i(T_i)$.

Formally, we can define the *path spaces* P_{xy} to be the space

$$\{\text{paths } \gamma : [a,b] \to \Lambda_K, \gamma(a) = x, \gamma(b) = y, \gamma^{(m)}(a) = \gamma^{(m)}(b) = 0 \text{ for all } m \text{ up to } M\}.$$

The derivative condition ensures that gluing two C^M paths results in another C^M path. Then $\alpha_1 \in P_{x_0 a_1(T_1)}$, $\alpha_i \in P_{a_{i-1}(0) a_i(T_i)}$, etc.



The idea is that a boundary of a J-holomorphic disk is naturally a Reeb string. In the usual definition of knot contact homology we count the J-holomorphic disks but never keep track of how the disks meet the Legendrians. Here we change that.

Given a J-holomorphic disk u, we define

$$\partial(u) = \beta_1 b_1 \cdots b_\ell \beta_{\ell+1}$$

where the b_i are the Reeb chords and β_i are the paths between chords.

Definition 2.6. We write

$$\mathcal{R}^\ell$$

to denote the space of Reeb strings with ℓ chords. $\mathcal{R}$ is the union of all the $\mathcal{R}^\ell$s, which is equipped with a notion of multiplication by concatenation. Finally, we write

$$C(\mathcal{R}) = \bigoplus C_d(\mathcal{R})$$

where $C_d(\mathcal{R})$ denotes the singular d-chains in $\mathcal{R}$.

How will we homologify this? We'll define two gradings on $C(\mathcal{R})$. The first comes from the singular chains:

(1) $C(\mathcal{R})$ has a grading d induces by *chain degrees*.
(2) There's also a boundary map ∂^{sing} which sends singular chains to their boundaries.
(3) The homology theory of $(C(\mathcal{R}), \partial^{sing})$ is the singular homology of the space $\mathcal{R}$.

From earlier, we also have degrees from the Morse indices (Maslov indices) of Reeb chords. This is the *chord degree*.

(1) A given singular chain $\sigma \in C(\mathcal{R})$ maps onto a finite number of Reeb chords $b_1, \ldots, b_\ell$. The *chord degree* of σ is

$$\sum |b_i|.$$

(2) More generally, a singular simplex $\sigma \in C(\mathcal{R})$ has a *type*, $\mathbf{a} = \alpha_1 a_1 \cdots a_\ell \alpha_{\ell+1}$ where the α_i are determined at least up to homotopy. Now for $u \in \mathcal{M}(a_i; \mathbf{b})$ we can define

$$\partial(u) \cdot_i \mathbf{a} = \alpha_1 a_1 \cdots \alpha_i \partial(u) \alpha_{i+1} \cdots a_\ell \alpha_{\ell+1}.$$

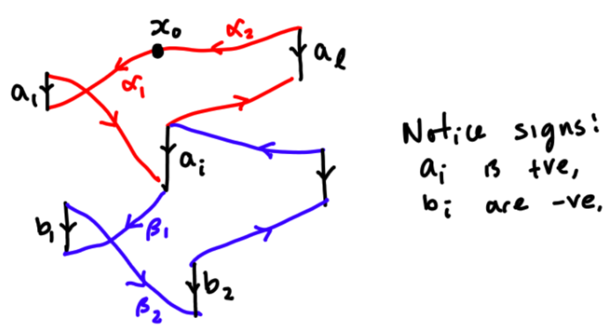
(The idea is that we're combining the string $\mathbf{a}$ with the string $\mathbf{b} = \partial(u)$ along their common Reeb chord.) Now we can define

$$\partial^{chord}(\mathbf{a}) = \sum_{i=1}^{\ell} \sum_{\substack{|a_i| - |\mathbf{b}| = 1 \\ u \in \mathcal{M}(a_i; \mathbf{b})/\mathbb{R}}} \varepsilon(u)(-1)^{\deg(\mathbf{a})} \partial(u) \cdot_i \mathbf{a}.$$

(3) Intuitively, instead of just counting holomorphic curves, we're keeping track of the information of the boundaries of each holomorphic curve.
(4) We only defined ∂^{chord} for simplices (since they have well defined Reeb string types), but this extends to chains.

$\underline{a} = \alpha_1 a_1 \cdots a_\ell \alpha_{\ell+1}$, $u \in \mathcal{M}(a_i; \underline{b})$

Then $\partial(u) \cdot_i \underline{a} = \alpha_1 a_1 \cdots \alpha_i \partial(u) \alpha_{i+1} \cdots a_\ell \alpha_{\ell+1}$



We're finally ready to define the boundary map $C(\mathcal{R}) \to C(\mathcal{R})$ that we'll use for our enhanced knot contact homology.

Theorem 2.7. *$(C(\mathcal{R}), \partial_\Lambda = \partial^{sing} + \partial^{chord})$ is a degree 1 differential on $C(\mathcal{R})$, where the degree of a chain in $C(\mathcal{R})$ is the sum of its singular and Reeb degrees. $\partial_\Lambda^2 = 0$, and the resulting homology theory (which we call knot contact homology) is a well defined knot invariant. (Specifically, it doesn't depend on the embedding of the knot, even though this influences Λ_K.)*

Example. Can we compute the knot contact homology of an unknot, embedded as an ellipse? There should only be two Reeb chords to deal with... but it seems messy and I didn't have enough time.

3. String homology recap

Last week we learned about string homology. Here's a very quick refresher.

Definition 3.1. Let K be embedded in $\mathbb{R}^3$. Fix a tubular neighbourhood NK of K. An *N-string* maps into NK, and a *Q-string* maps into $\mathbb{R}^3$. A *broken string with* 2ℓ *switches* is a collection of such strings, $NQNQ \cdots QN$, where each N and Q string starts and ends on K, passing from an N string to a Q string is smooth, and passing from a Q string to an N string is anti-smooth.

Definition 3.2. We let $C_0(\Sigma^\ell)$ denote the free $\mathbb{Z}$-module generated by broken strings with 2ℓ switches. We let $C_1(\Sigma^\ell)$ denote the free $\mathbb{Z}$-module of homotopy classes of broken strings with 2ℓ switches.

Next, we need a boundary map $C_1(\Sigma^\ell) \to C_0(\Sigma^\ell)$. One possible boundary map is obvious: it corresponds to the actual boundary,

$$\partial(s^\lambda) = s^1 - s^0$$

where s^λ is a homotopy of broken strings. Another boundary map is a little more interesting: from last week, the second boundary map corresponds to *inserting spikes* whereever our broken string meets the knot, in a way that produces another broken string.

More precisely, $\delta(s^\lambda) = \sum_i \varepsilon^i(s^{\lambda_i} \cdot_j \mathfrak{s})$ where we've used our notation from contact homology to insert the spike $\mathfrak{s}$.

Notice that $\partial : C_1(\Sigma^\ell) \to C_0(\Sigma^\ell)$ while $\delta : C_1(\Sigma^\ell) \to C_0(\Sigma^{\ell+1})$. We define

$$H_0^{string}(K) = H_0(\Sigma) = C_0(\Sigma)/\operatorname{im}(\partial + \delta)$$

where $C_i(\Sigma) = \bigoplus C_i(\Sigma^\ell)$.

Theorem 3.3. *$H_0^{contact}(K) = H_0^{string}(K)$.*

We won't prove this, but we'll describe the chain map from the knot contact complex to the string complex.

4. The chain map from knots to strings

Let (C_i, ∂) and (D_i, δ) be chain complexes. A *chain map* $\varphi_i : C_i \to D_i$ is a map satisfying

$$\varphi_{i-1}\partial_i = \delta_i \partial_i.$$

That is, it's a map which respects the degrees of the chain complexes and commutes with the boundary maps. This is exactly the required property to induce a map on homology groups.

Proposition 4.1. A chain map $\varphi : C \to D$ induces maps $\varphi_* : H(C) \to H(D)$.

Therefore we'll define a chain map

$$(C^{contact}(K), \partial_\Lambda) \to (C^{string}(K), \partial + \delta).$$

Loosely speaking, the *paths* in $C^{contact}(K)$ correspond to N-strings, while *chords* correspond to Q-strings. At the level of boundary maps, ∂^{sing} corresponds to ∂, and ∂^{chord} corresponds to δ.

(1) For each ℓ, we define $\mathcal{M}_\ell(a)$ to consist of J-holomorphic curves with a positive marked point mapping onto a, and ℓ extra marked points with fixed local winding number 1/2. Unforutnately I'm not really sure what this means (or how to draw it). We can compactify the Moduli space, and write $\overline{\mathcal{M}_\ell}(a)$. This space has dimension $|a|$. (That is, it's independent of ℓ.)

(2) Given a Reeb chord a, we define $\Phi_\ell(a) : \overline{\mathcal{M}_\ell}(a) \to \Sigma^\ell$ to send a J-holomorphic curve from a to $\mathbf{b}$ to the broken string in Σ^ℓ, where the path from a to b_1 is the first N-string, then b_1 is the first Q-string, the path from b_1 to b_2 is the second N-string, and so on. We interpret $\Phi_\ell(a)$ as a singular chain (by decomposing $\overline{\mathcal{M}_\ell}(a)$ into a triangulation, i.e. a union of simplices). Thus we have a map

$$\Phi_\ell : \{\text{Reeb chords}\} \to C(\Sigma^\ell).$$

(3) The key identity we need is that

$$\Phi_\ell(\partial^{chord} a) = \partial \Phi_\ell(a) + \delta \Phi_{\ell-1}(a).$$

(4) Next we define $\Phi(a) = \sum_{\ell=0}^{\infty} \Phi_\ell(a) \in C(\Sigma)$. (In fact, the sum to ∞ is shown in the paper to be a finite sum!

(5) Finally, we extend to simplices σ in $C(\mathcal{R})$. Suppose σ has type

$$\alpha_1 a_1 \cdots \alpha_{ell} a_\ell \alpha_{\ell+1}.$$

Then $\Phi(\sigma) = \alpha_1 \Phi(a_1) \cdots \alpha_\ell \Phi(a_\ell) \alpha_{\ell+1}$.

(6) In each step, the way we've extended the map ensures that the key identity also generalises. The last step makes ∂^{sing} non-trivial, but we end up with

$$\Phi(\partial^{sing}\sigma) + \Phi(\partial^{chord}\sigma) = \partial\Phi(\sigma) + \delta\Phi(\sigma).$$