GROMOV-WITTEN INVARIANTS AND SW=GR

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Abstract. This is the 3rd installment in a series of talks concerning Taubes's result on the equality of Seiberg-Witten and Gromov-Witten invariants. The first two talks focused on defining the Seiberg-Witten invariants. In this talk, I'll start by providing some applications of the Seiberg-Witten invariants so that we know why we bothered to define them to begin with. Next I'll describe the Gromov-Witten invariants together with some applications, and we'll finish with an overview of the equality of the two invariants.

1. Seiberg-Witten invariants: recap of definition

Let X be a closed smooth oriented 4-manifold. Then X admits a *spin^c structure*, i.e. a lift of the frame bundle (a principal $SO(4)$-bundle) to a principal $\mathrm{Spin}^c(4)$-bundle. We write $\mathrm{Spin}^c(X)$ to denote the collection of spin^c structures on X. Recall that

$$\mathrm{Spin}^c(4) = U(1) \times \mathrm{Spin}(4)/\pm 1,$$

so there is a natural projection $\det : \mathrm{Spin}^c(4) \to U(1)$ which looks like the squaring double cover. This map determines a complex line bundle $\mathcal{L} \to X$, called the determinant line bundle of $s \in \mathrm{Spin}^c(X)$. Finally we note that any $s \in \mathrm{Spin}^c(X)$ can equivalently be considered as two principal $U(2)$-bundles $S^{\pm} \to X$, with an additional *compatibility condition* $\gamma : TX \to \mathrm{End}(S^+ \oplus S^-)$.

Our objects of interest are pairs (A, φ), where A is a $U(1)$-connection on $\mathcal{L}$, and $\varphi \in \Gamma(S^+)$ is a *self-dual spinor field.* (A *$U(1)$-connection on $\mathcal{L}$* is an operator $d_A : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$ which locally looks like like $d_A\sigma = d\sigma + A\sigma$, where $A \in i\Omega^1(\mathcal{L})$.) The idea of the Seiberg-Witten invariants is that they measure how many solutions (A, φ) exist given some non-linear PDEs called the Seiberg-Witten equation (up to gauge).

Specifically, the Seiberg-Witten equations are

$$D^A\varphi = 0, \quad F_A^+ = \sigma(\varphi)$$

where D^A is the Dirac operator corresponding to A, and $F_A^+ = (F_A + \star F_A)/2$ is the self-dual part of the curvature 2-form of A, and σ is a squaring map.

The Seiberg-Witten equations are invariant under the action of the gauge group $\mathcal{G} = \{g : X \to \mathbb{S}^1\}$. We define the *moduli space of solutions to the Seiberg-Witten equations* $\mathcal{M}(s)$ to be

$$\{(A, \varphi) \text{ solving the SW equations}\}/\mathcal{G}.$$

Proposition 1.1. Suppose $b_2^+ \geq 2$. Then $\mathcal{M}(s)$ has no reducible solutions, and is hence a manifold. Moreover, it is either empty or has dimension

$$\dim \mathcal{M}(s) = \frac{c_1(s)^2 - 2\chi(X) - 3\sigma(X)}{4}.$$

If X is symplectic, then $\mathcal{M}(s)$ is empty or zero dimensional! Since $\mathcal{M}(s)$ is oriented, there is a well defined signed count $SW_X(s)$ of solutions to the SW equations.

Definition 1.2. The *Seiberg-Witten invariant* is the map

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$$

defined as the signed count of $\mathcal{M}(s)$ for each $s \in \mathrm{Spin}^c(X)$.

For closed oriented 4-manifolds, if $H^2(X;\mathbb{Z})$ has no 2-torsion, there is a canonical identification

$$\mathrm{Spin}^c(X) \xrightarrow{c_1} \mathrm{Char}(X) \subset H^2(X;\mathbb{Z}),$$

where $\mathrm{Char}(X) = \{a \in H^2(X;\mathbb{Z}) : \langle a, b\rangle \equiv \langle b, b\rangle \mod 2 \text{ for all } b \in H^2(X;\mathbb{Z})\}$. Therefore we can define the SW invariant to be a map

$$SW_X : \mathrm{Char}(X) \to \mathbb{Z}.$$

Definition 1.3. We noted that symplectic 4-manifolds satisfy the useful property that $\mathcal{M}(s)$ is either empty or 0 dimensional. In general, a 4-manifold satisfying this property is said to be of *simple type*. It is open whether or not all manifolds are of simple type.

2. Seiberg-Witten invariants: applications

One of the great applications is that it gives comparitively easy proofs of the existence of exotic smooth structures on 4-manifolds. We start by listing a few properties of the Seiberg-Witten invariants. X is assumed to be a closed oriented 4-manifold of simple type, with $b_2^+ \geq 2$, and $H^2(X;\mathbb{Z})$ having no 2-torsion. Thus the Seiberg-Witten invariant is a well defined map $SW : \mathrm{Char}(X) \to \mathbb{Z}$.

(1) $SW_X(k)$ vanishes for all but finitely many $k \in \mathrm{Char}(X)$. The $\{k_1, \dots, k_s\}$ for which $SW_X(k) \neq 0$ are called *basic classes*.
(2) Suppose $X = X_1 \# X_2$, where $b_2^+(X_i) \geq 1$ for both i. Then SW_X vanishes identically.
(3) (Blow up formula.) Suppose $X' = X \# \overline{\mathbb{CP}^2}$, and let $\{k_1, \dots, k_s\}$ denote the basic classes of X. Let E be the class of $\overline{\mathbb{CP}^1}$ in $\overline{\mathbb{CP}^2}$. Then $\{k_i \pm E\}$ are the basic classes of X', and $SW_{X'}(k_i \pm E) = \pm SW_X(k_i)$.
(4) Suppose X is a complex projective surface. Then $c_1(TX)$ is a characteristic element, and

$$SW_X(\pm c_1(TX)) = \pm 1.$$

Theorem 2.1. *There exist (simply connected) closed oriented 4-manifolds which are homeomorphic but not diffeomorphic.*

Proof. Let $X_1 = K3\#\overline{\mathbb{CP}^2}$, and $X_2 = 3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$. These are connected sums of simply connected manifolds, and hence simply connected. By Freedman's classification of simply connected 4-manifolds, they are homeomorphic if they have equivalent intersection forms

$$Q_{X_i} : H^2(X_i;\mathbb{Z}) \otimes H^2(X_i;\mathbb{Z}) \to \mathbb{Z}.$$

We have

$$Q_{X_1} = 2(-E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (-1), \quad Q_{X_2} = 3(1) \oplus 20(-1).$$

These can easily be shown to be equivalent using the classification of unimodular bilinear forms. Alternatively, one can directly prove this by finding the appropriate change of basis. Therefore X_1 and X_2 are homeomorphic.

On the other hand, by property 4 of the SW invariants, since $K3$ is a complex projective surface, $SW_{K3}(c_1(TK3)) = 1$. By property 3,

$$SW_{X_1}(c_1(TK3) + E) = SW_{K3}(c_1(TK3)) = 1.$$

In particular, SW_{X_1} is not identically vanishing.

However, we can write X_2 as $(2\mathbb{CP}^2\#20\overline{\mathbb{CP}^2})\#(\mathbb{CP}^2)$. Then both sides of the connected sum have $b_2^+ \geq 1$, so SW_{X_2} vanishes identically by property 2. Therefore

$$SW_{X_1} \neq SW_{X_2},$$

so X_1 and X_2 are not diffeomorphic. □

3. Gromov-Witten invariants: definition

Loosely speaking, the Gromov-Witten invariants are a count of isolated J-holomorphic curves in a symplectic manifold. Several weeks ago Sarah introduced the "symplectic geometry from the J-holomorphic perspective". I will briefly recount some of the relevant definitions.

Definition 3.1. For $(M, j), (N, J)$ almost complex, the *Cauchy-Riemann equation* for a map $f : M \to N$ is

$$df \circ j = J \circ df.$$

If f satisfies the Cauchy-Riemann equation, it is said to be *(j,J)-holomorphic.*

Remark. The above condition is exactly the requirement that f preserves the complex structures. As an exercise, one can compute the standard Cauchy-Riemann equations for $f : \mathbb{C} \to \mathbb{C}$ from this version.

To make things even more concise, we can introduce the holomorphic and antiholomorphic differentials:

Definition 3.2. Given $f : (M, j) \to (N, J)$, the *holomorphic* and *antiholomorphic* differentials are given by

$$\partial_J f = \frac{1}{2}(df - J \circ df \circ j), \quad \overline{\partial}_J f = \frac{1}{2}(df + J \circ df \circ j).$$

Remark. Notice that $d = \partial_J + \overline{\partial_J}$. Moreover, f is holomorphic if and only if $\overline{\partial_J} f = 0$.

Proposition 3.3. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is said to be *compatible* if

$$\omega = J^*\omega, \quad \omega(v, Jv) > 0 \text{ whenever } v \neq 0.$$

Given any symplectic manifold, there is a contractible family of compatible complex structures on it, which we denote by $\mathcal{J}(M, \omega)$.

Definition 3.4. Let (M, ω) be a symplectic manifold, and $J \in \mathcal{J}(M, \omega)$. Let (Σ, j) be a Riemann surface. A map $u : \Sigma \to M$ is a *J-holomorphic curve* if $\overline{\partial}_J u = 0$.

Definition 3.5. Let (M, ω) be a symplectic manifold, and $J \in \mathcal{J}(M, \omega)$. Fix a homology class $A \in H_2(M; \mathbb{Z})$, and a Riemann surface (Σ, j). We write

$$\mathcal{M}(A, \Sigma; J) \coloneqq \{u \in C^\infty(\Sigma, M) : [u] = A, \overline{\partial}_J u = 0\}/\sim$$

where $u \sim u'$ if they're isomorphic. (That is, there is a biholomorphic map $\varphi : \Sigma \to \Sigma$ such that $u = u' \circ \varphi$.)

A little more generally, we can un-fix the choice of Riemann surface, and simply consider maps from arbitrary Riemann surfaces with a prescribed genus.

Definition 3.6. Given (M, ω) and $J \in \mathcal{J}(M, \omega)$, $A \in H_2(M; \mathbb{Z})$, and an integer $g \geq 0$,

$$\mathcal{M}_{g,0}(A; J) \coloneqq \{u : (\Sigma_g, j) \to M : [u] = A, \overline{\partial}_J u = 0\}/\sim.$$

Finally we can also add *marked points*, which are an ordered tuple of distinct points in the domain.

Definition 3.7. Given (M, ω, J), $A \in H_2(M; \mathbb{Z})$, and $g, m \geq 0$, the *moduli space of J-holomorphic curves in M with genus g and m marked points representing A* is

$$\mathcal{M}_{g,m}(A; J) \coloneqq \{(\Sigma, j, u, (z_1, \ldots, z_m))\}/\sim,$$

where $(\Sigma, j, u, \Theta) \sim (\Sigma', j', u', \Theta')$ if there is a biholomorphism $\varphi : \Sigma \to \Sigma'$ such that $u = u' \circ \varphi$, and $\varphi(\Theta) = \Theta'$ (with orders preserved).

As mentioned at the start of this section, the goal of the Gromov-Witten invariants is to "count the number of curves" in $\mathcal{M}_{g,m}(A; J)$. For this we want the space to be a compact smooth manifold.

Theorem 3.8. *Let M be $2n$-real-dimensional. For generic J, $\mathcal{M}_{g,m}(A; J)$ is a smooth manifold of dimension*

$$\dim M_{g,m}(A; J) = (n - 3)(2 - 2g) + 2c_1(A) + 2m.$$

All we need now is a compactness result, but this also holds by theorem that Sarah explained a few weeks ago:

Theorem 3.9 (Gromov compactness)**.** *The space $\overline{\mathcal{M}}_{g,m}(A; J)$ is compact.*

Here $\overline{\mathcal{M}}_{g,m}(A;J)$ is obtained from $\mathcal{M}_{g,m}(A;J)$ by gluing J-holomorphic bubble trees. More formally, we can define

$$\mathcal{C}(A;J) \coloneqq \{u : (\Sigma, j) \to M : [u] = A, \overline{\partial_J} u = 0\},$$

where $u : (\Sigma, j) \to M$ is a *nodal curve*. Then the closure of $\mathcal{M}_{g,m}(A;J)$ in $\mathcal{C}(A;J)$ is compact (and the boundary points are well understood: they consist only of finite spherical bubble trees).

By establishing that $\overline{\mathcal{M}}_{g,m}(A;J)$ is a smooth compact manifold, we can proceed to try and make some invariants!

The first observation is that the m marked points of $\overline{\mathcal{M}}_{g,m}(A;J)$ provide some natural evaluation maps: for each i, we define

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,m}(A;J) \to M$$

by

$$(\Sigma, j, u, (z_1, \ldots, z_m)) \mapsto u(z_i).$$

There's another map

$$\pi : \overline{\mathcal{M}}_{g,m}(A;J) \to \overline{\mathcal{M}}_{g,m}$$

defined to be a forgetful map. Here

$$\overline{\mathcal{M}}_{g,m} \coloneqq \{u : (\Sigma_g, j) \to \{\mathrm{pt}\} : [u] = 0,\ m \text{ marked points}\}/\sim = \overline{\mathcal{M}}_{g,m}(\{\mathrm{pt}\}, 0; J).$$

The forgetful map is given by $u \mapsto p \circ u$, where $p : M \to \{\mathrm{pt}\}$. We now have a diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,m}(A;J) & \xrightarrow{(\mathrm{ev}_1,\ldots,\mathrm{ev}_m)} & M \times \cdots \times M \\ \Big\downarrow \pi & & \\ \overline{\mathcal{M}}_{g,m}. & & \end{array}$$

Definition 3.10. The *Gromov-Witten invariants* are the homomorphisms

$$GW^M_{g,m,A} : H^*(M;\mathbb{Q})^{\otimes m} \otimes H_*(\overline{\mathcal{M}}_{g,m};\mathbb{Q}) \to \mathbb{Q}$$

defined by

$$GW^M_{g,m,A}(a_1, \ldots, a_m; \beta) \coloneqq \int_{\overline{\mathcal{M}}_{g,m}(A;J)} \mathrm{ev}_1^* a_1 \smile \cdots \smile \mathrm{ev}_m^* a_m \smile \pi^* PD(\beta),$$

for (M, ω) symplectic with compatible almost complex structure J, $A \in H_2(M;\mathbb{Z})$, and $g, m \geq 0$.

This is nonsense! Let's try to at least interpret it somewhat. For each i, let X_i be Poincaré dual to a_i. Then $\mathrm{ev}_i^* a_i$ is dual to the collection of curves u with $u(z_i) \in X_i$ (which is a space in $\overline{\mathcal{M}}_{g,m}(A;J)$. The cup product is Poincaré dual to intersections, so $\mathrm{ev}_1^* a_1 \smile \cdots \smile \mathrm{ev}_m^* a_m$ represents all the maps with $u(z_i) \in X_i$ for each i. Finally by cupping this with $\pi^* PD(\beta)$, we restrict our collection to those curves which represent β apon projection to $\overline{\mathcal{M}}_{g,k}$. The integral essentially counts the collection of all such curves.

By playing around with g, m, and A, we have a vast number of Gromov-Witten invariants for a given symplectic manifold.

Why do we care about Gromov-Witten invariants?

(1) The values of the invariants themselves is interesting - in the setting of algebraic geometry (to which Gromov-Witten invariants can be transported), counting curves is a classical difficult problem which is formalised by the Gromov-Witten invariants. The Gromov-Witten invariants are difficult to compute, but *properties* of the invariants can be decuded which aid in solving these problems.
(2) Defining the invariants is technical and difficult, but these issues turned out to be the same ones that arise in other settings such as defining Hamiltonian Floer homology (which was used to solve the Arnold conjecture). This subsequently gave rise to other Floer homologies which have been very useful in e.g. low dimensional topology.
(3) Mirror symmetry!
(4) Distinguishing previously indistinguishable symplectic manifolds. (This application is actually a bit fake, since the invariants are very difficult to compute so it hasn't been as successful in this realm as one might expect.)

4. The statement $SW = Gr$

The main result of Taubes is that the *Seiberg-Witten invariants* for a symplectic 4-manifold X with $b_2^+ \geq 2$ are equivalent to its *Gromov-Witten invariants*. Formally, the theorem states that

$$SW_X(K_X^* + 2\varepsilon) = Gr(\varepsilon).$$

In the rest of this section we'll interpret what this actually means.

On the left, we've written $SW_X(K_X^* + 2\varepsilon)$. Since X is symplectic, it admits a compatible almost complex structure J. This determines a canonical spin^c structure s_J. Recall that we have a map $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$. It turns out that $c_1(s_J) \in H^2(X; \mathbb{Z})$ is independent of the choice of compatible almost complex structure J! It only depends on (X, ω), and we denote it by K_X^* This is actually the chern class of the *anti-canonical bundle*. (I.e. the dual of the top exterior power of the complex vector bundle (TX, J).)

Any $s \in \mathrm{Spin}^c(X)$ can now be written in the form $s_J + \varepsilon$, so $c_1(s_J + \varepsilon) = K_X^* + 2\varepsilon \in H^2(X; \mathbb{Z})$ parametrises all of the spin^c structures. Interpreting the Seiberg-Witten invariant as a map

$$SW_X : \mathrm{Char}(X) \to \mathbb{Z},$$

everyting in $\mathrm{Char}(X)$ can be written as $K_X^* + 2\varepsilon$.

On the right, we've written $Gr(\varepsilon)$. This is called the *Gromov invariant* and is a specific version of the Gromov-Witten invariant. In Gromov-Witten theory, we consider only connected curves, but the Gromov invariant allows for disconnected curves.

Since $\varepsilon \in H^2(X; \mathbb{Z})$, by Poincaré duality, it determines a homology class $A_\varepsilon \in H_2(X; \mathbb{Z})$. We define $\mathcal{G}(A_\varepsilon; J)$ to be the moduli space of all (possible disconnected) J-holomorphic curves representing the class A. One can show that for J generic, $\mathcal{G}(A_\varepsilon; J)$ is a compact smooth manifold with

$$\dim \mathcal{G}(A;J) = K_X^* \cdot A_\varepsilon + A_\varepsilon \cdot A_\varepsilon.$$

This is always even dimensional; we can write $2d = \dim \mathcal{G}(A;J)$. To carry out a signed count we want to reduce the dimension to 0 - which we do by adding d marked points.

We define the *Gromov invariant* to be

$$Gr(\varepsilon) = \#\mathcal{G}_{0,d}(A;J).$$

5. A proof outline of $SW = Gr$ in a special case

The full proof of this theorem takes 400 pages of analysis and is definitely beyond the scope of this talk. The idea of the proof is to deform the Seiberg-Witten equations so they get "close" to a Cauchy-Riemann operator on the line bundle of the Chern class ε. Solutions to the Seiberg-Witten equations then correspond to an almost-holomorphic section of the line bundle, and the zero set of the section is a J-holomorphic curve representing ε.

However, an important preliminary result connecting symplectic geometry and 4-dimensional topology is the following:

Theorem 5.1. *Let X be a symplectic manifold with $b_2^+ \geq 2$. Then*

$$SW_X(\pm K_X^*) = \pm 1.$$

Proof idea. We can perturb the Seiberg-Witten equations

$$\mathcal{D}^A\varphi = 0, \quad F_A^+ = \sigma(\varphi)$$

to look like

$$\mathcal{D}^A\varphi = 0, \quad F_A^+ - F_{A_0}^+ = \sigma(\varphi) - \rho^2\omega$$

where A_0 is a special connection satisfying some properties and ρ is a parameter. The existnce of A_0 satisfying prescribed properties depends on X being symplectic. There's a bijection between solutions of the Seiberg-Witten equations and the perturbed version, but in this version as we take ρ to infinity, there's a unique solution. □