$\chi(\mathcal{K}) \leq \langle c_1(\omega), [\mathcal{K}]\rangle - \mathcal{K}\cdot\mathcal{K}$

A combinatorial proof of the adjunction inequality

An expository poster on work by Peter Lambert-Cole

What is the adjunction inequality?

Low dimensional topology was revolutionised in the 80s and 90s by the introduction of gauge theory. Notably, gauge theory was used to demonstrate the first examples of exotic smooth structures on some 4-manifolds, as well as the first solutions to some minimal genus problems for embedded surfaces. The adjunction inequality is an example of the latter.

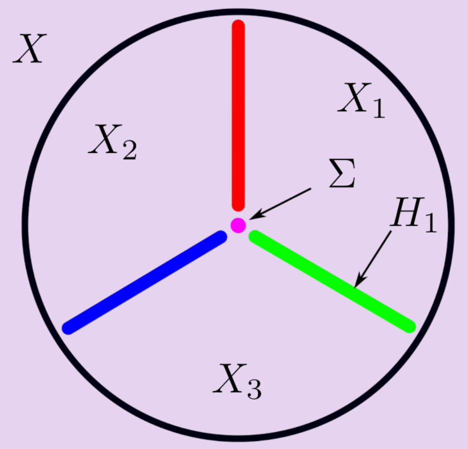
Fix a closed symplectic manifold (X, ω) with $[\omega]$ integral. If $\mathcal{K} \subset X$ is a smoothly embedded surface with positive sympletic area, then

$$2g(\mathcal{K}) - 2 \geq \mathcal{K}\cdot\mathcal{K} - \langle c_1(\omega), [\mathcal{K}]\rangle.$$

Intuitively the homological information expressed on the right side of the inequality provides a lower bound on the genus of $\mathcal{K}$. This is just one example of an adjunction inequality, as stated in [PLC2]. Using Seiberg-Witten gauge theory, there is an adjunction inequality for surfaces in closed smooth 4-manifolds with non-trivial Seiberg-Witten invariant (which need not be symplectic).

What are trisections?

Trisections are a 4-dimensional analogue to Heegaard splittings of 3-manifolds. Specifically, if X is a closed smooth 4-manifold, a trisection of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ where each X_i is a 4-dimensional 1-handlebody, each $H_i = X_i \cap X_{i-1}$ is a 3-dimensional 1-handlebody, and $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed surface and the boundary of each H_i.



The big idea is that trisections are determined by their *spines*, i.e. the union of the H_i. This allows 4-dimensional manifolds to be encoded in lower dimensions. The hope now is that techniques from algebraic and geometric topology (i.e. "combinatorial" techniques) will be sufficient to reprove results that initially required gauge theory.

In our context *Weinstein trisections* are used, which are trisections whose three sectors are all Weinstein domains.

Surfaces in bridge position

Let $\mathcal{K} \subset X$ be a surface embedded in a trisected 4-manifold. If $\mathcal{K}$ is in *bridge position* it is determined by its intersection with the spine - in much the same way that X is determined by its spine. $\mathcal{K}$ is said to be in *bridge position* if

- its intersection with each X_i is a disjoint union of disks which can be isotoped to simultaneously lie on ∂X_i (i.e. a trivial disk tangle), and
- its intersection with each H_i is a disjoint union of arcs which can be isotoped to simultaneously lie on ∂H_i (i.e. a trivial tangle).

Moreover, its genus is given by $2 - 2g(\mathcal{K}) = \chi(\mathcal{K}) = c_1 + c_2 + c_3 - b$ where b is half the number of intersection points in $\mathcal{K} \cap \Sigma$, and each c_i is the number of components in $\mathcal{K} \cap \partial X_i$. This is the first stepping stone for proving an adjunction inequality using only combinatorial techniques.

The proof ingredients

1. Finding a trisection

It turns out that all closed symplectic 4-manifolds admit Weinstein trisections! Auroux showed that any closed symplectic 4-manifold X admits a branched cover

$$X \to \mathbb{CP}^2.$$

The branch locus $\mathcal{R} \subset \mathbb{CP}^2$ can be isotoped to be in *transverse bridge position* in a given Weinstein trisection of $\mathbb{CP}^2$, and a Weinstein trisection of X is then induced via pullback. (Roughly speaking a surface is in *transverse bridge position* if it is in bridge position and the $\mathcal{K} \cap \partial X_i$ are transverse links. Note that ∂X_i is a contact manifold.)

$\mathbb{CP}^2$ admits a remarkably simple Weinstein trisection: the *standard trisection* of $\mathbb{CP}^2$ has sectors homeomorphic to 4-balls, and a spine formed by gluing solid tori to a central torus $\Sigma = \mathbb{T}^2$.

2. Transverse bridge case

Let us assume $\mathcal{K} \subset X$ is in transverse bridge position with respect to a Weinstein trisection. Each $K_i = \mathcal{K} \cap \partial X_i$ is a transverse link, and has a well defined self-linking number $sl(K_i)$.

By functoriality, the first Chern class $c_1(\omega)$ is the pullback of the first Chern class of the Fubini-Study form ω_{FS} of $\mathbb{CP}^2$. This provides a Poincaré dual whose intersections with the trisection can be understood. It can then be shown that

$$sl(K_1) + sl(K_2) + sl(K_3) = \mathcal{K}\cdot\mathcal{K} - \langle c_1(\omega), [\mathcal{K}]\rangle - b$$

(where $b = \frac{1}{2}\#\mathcal{K} \cap \Sigma$.) The adjunction inequality now follows from the last identity in ingredient 4 and the slice-Bennequin inequality.

3. General case: isotoping

Generally an embedded surface $\mathcal{K} \subset X$ is not in transverse bridge position, nor is it isotopic to a surface in transverse bridge position. However, it *is* possible to isotope $\mathcal{K}$ so that the tangles $\mathcal{K} \cap H_i$ are homotopic in H_i to positively transverse tangles (provided $\mathcal{K}$ has positive symplectic area). Our strategy is now:

(a) Isotope $\mathcal{K}$ to a *homotopically transverse* surface $\mathcal{K}'$ as above. The terms in the adjunction inequality are isotopy invariant.

(b) Homotope $\mathcal{K}'$ to an immersed surface $\mathcal{L}$ in transverse bridge position. Trace how the terms of interest change from $\mathcal{K}'$ to $\mathcal{L}$.

Showing that $\mathcal{K}$ can be isotoped to be homotopically transverse requires studying the fundamental groups of the H_i.

4. General case: homotoping

Let $\mathcal{K}' \subset X$ be in homotopically transverse bridge position. The links K_i are homotoped to positively transverse links $L_i \subset \partial X_i$. Their unions over i are links $K, L \subset Y = \sqcup_i \partial X_i$. The homotopy $K \sim L$ is encoded by $2n$ crossing changes (lifted from n crossing changes in the H_i). These are surgically resolved by removing two 4-balls from each crossing pair in Y, and gluing in $[0,1] \times \mathbb{S}^3$. Links $\widetilde{K}, \widetilde{L} \subset \widetilde{Y}$ are induced from K and L by adding $2n$ bands: two for each $[0,1] \times \mathbb{S}^3$. From the earlier special case we have

$$sl(L) = \mathcal{K}\cdot\mathcal{K} - \langle c_1(\omega), \mathcal{K}\rangle - b.$$

Adding bands changes the self linking number to

$$sl(\widetilde{L}) = \mathcal{K}\cdot\mathcal{K} - \langle c_1(\omega), \mathcal{K}\rangle - b + 2n.$$

$\widetilde{K}$ and $\widetilde{L}$ are isotopic, and $\widetilde{K}$ bounds a ribbon surface F satisfying

$$\chi(F) = c_1 + c_2 + c_3 - 2n.$$

5. Slice-Bennequin inequality

Rasmussen's invariant provided the first combinatorial proof of the slice-Bennequin inequality for $\mathbb{S}^3$: given a transverse link $\widetilde{L} \subset \mathbb{S}^3$, if $F \subset B^4$ is a smoothly embedded surface with boundary $\widetilde{L}$, then

$$sl(\widetilde{L}) \leq -\chi(F).$$

A combinatorial proof of the above inequality for links in $\#_k \mathbb{S}^1 \times \mathbb{S}^2$ is given in [PLC]. Combining this inequality with those from the previous proof ingredient gives

$$\mathcal{K}\cdot\mathcal{K} - \langle c_1(\omega), \mathcal{K}\rangle - b + 2n \leq 2n - c_1 - c_2 - c_3.$$

Substituting the bridge position formula for $\chi(\mathcal{K})$ finishes the proof!

References

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