# LEGENDRIAN KNOTS AND MONOPOLES II

SHINTARO FUSHIDA-HARDY

ABSTRACT. We give a proof outline of Mrowka and Rollin's general slice-Bennequin inequality. Specifically, this uses a version of Seiberg-Witten theory for 4-manifolds with contact boundary. They prove an excision result in this framework to extend the adjunction inequality to surfaces with boundary.

## 1. THE ADJUNCTION INEQUALITY

The general form of the slice-Bennequin inequality that we'll be proving makes use of the adjunction inequality. The statement of the slice-Bennequin inequality is also very similar to the adjunction inequality (but only more complicated), so we'll begin by introducing the adjunction inequality.

**Theorem 1.1.** *Let $M$ be a 4-manifold and $\mathfrak{s} \in \mathrm{Spin}^c(M)$ such that $SW_M(\mathfrak{s})$ is non-zero. Then any closed surface $\Sigma \subset M$ with $[\Sigma] \neq 0$ and $[\Sigma]^2 \geq 0$ satisfies*

$$-\chi(\Sigma) \geq [\Sigma]^2 + |c_1(\mathfrak{s}) \cdot [\Sigma]|.$$

We'll run through a bullet list to make sure the theorem makes sense:

- Given a manifold $M$, spin$^c$ structures are like "lifts" of oriented orthonormal frame bundles of $M$. Specifically, the $\mathrm{SO}(n)$ structure is upgraded to a $\mathrm{Spin}^c(n)$ structure. Since this is a bundle, it has characteristic classes - we have a map

$$c_1 : \mathfrak{s} \mapsto c_1(\mathfrak{s}) \in H^2(M; \mathbb{Z}).$$

- Given a spin$^c$ structure $\mathfrak{s}$, we can consider certain differential equations on $M$. The signed count of solutions to the equations is $SW_M(\mathfrak{s})$.
- Both terms on the right side of the equation are integer valued, so we can make sense of the inequality (which provides a lower bound on the genus of $\Sigma$).

The adjunction inequality can also be stated for manifolds with boundary - but there are a couple of things we must be careful about:

- When the base manifold $\overline{M}$ has boundary, rather than considering $\mathrm{Spin}^c(\overline{M})$, we want some notion of relative spin$^c$ structures.
- Earlier we define the Seiberg-Witten invariant for ordinary spin$^c$ structures, but we'll need to extend it to relative spin$^c$ structures.

What exactly are our relative spin$^c$ structures going to be?

Typically the reason we need relative versions of mathematical objects over manifolds with boundary is because "the differences above the boundary get lost" unless we explicitly keep track of boundary conditions. Relative spin$^c$ structures are the same.

- Let $\overline{M}$ be a 4-manifold with boundary, and $Y = \partial\overline{M}$ a contact manifold (with contact form $\xi$). Then $(Y, \xi)$ has a canonical spin$^c$ structure $\mathfrak{s}_\xi$. This is because in 3-dimensions spin$^c$ structures correspond to Hermitian bundles of rank 2, and the contact distribution determines such a bundle.
- The space of relative spin$^c$ structures, $\text{Spin}^c(\overline{M}, \xi)$, consists of pairs

$$(\mathfrak{s}, h)$$

where $\mathfrak{s}$ is a spin$^c$ structure over $\overline{M}$, and $h$ is an isomorphism $h : \mathfrak{s}|_Y \to \mathfrak{s}_\xi$.
- The main idea is that carrying the data of $h$ refines what $\mathfrak{s}$ is doing over the boundary.

We can now state the adjunction inequality for closed surfaces in 4-manifolds with contact boundary:

**Theorem 1.2.** *Let $\overline{M}$ be a 4-manifold with contact boundary $(Y, \xi)$, and $(\mathfrak{s}, h) \in \text{Spin}^c(\overline{M}, \xi)$ such that $SW_{\overline{M},\xi}(\mathfrak{s}, h) \neq 0$. Then any closed surface $\Sigma$ with $g(\Sigma) \geq 1$ and $[\Sigma]^2 = 0$ satisfies*

$$-\chi(\Sigma) \geq |c_1(\mathfrak{s}) \cdot [\Sigma]|.$$

To really understand this theorem, we should know that the relative version of Seiberg-Witten invariants are. Unfortunately I missed some earlier talks this quarter, but my understanding is that these have already been defined: essentially

$$SW : \text{Spin}^c(\overline{M}, \xi) \to \mathbb{Z}$$

is a count of solutions to differential equations over $\overline{M}$ with some boundary conditions.

We'll make use of this version of the adjunction inequality to prove the slice-Bennequin inequality.

## 2. The slice-Bennequin inequality

The slice Bennequin inequality is essentially a version of the adjunction inequality for surfaces with boundary. The statement is very similar, but the homological invariants in the inequality must be made relative in some way.

**Theorem 2.1.** *Let $\overline{M}$ be a 4-manifold with contact boundary $(Y, \xi)$, and $(\mathfrak{s}, h) \in \text{Spin}^c(\overline{M}, \xi)$ such that $SW_{\overline{M},\xi}(\mathfrak{s}, h) \neq 0$. Let $\Sigma \subset \overline{M}$ be a surface with boundary a Legendrian knot $K \subset Y$. Then*

$$-\chi(\Sigma) \geq \text{tb}(K, [\Sigma]) + |\,\text{rot}(K, [\Sigma], (\mathfrak{s}, h))|.$$

- The Thurston Bennequin invariant $\text{tb}(K, [\Sigma])$ is analogous to the term $[\Sigma]^2$ in the adjunction inequality. It's defined to be the linking number of the relative homology class $[\Sigma] \in H_2(\overline{M}, Y; \mathbb{Z})$ with itself, given the boundary knot $K$ is pushed off transversely to the contact distribution in $Y$. (In the 4-ball case, which is the traditional setting for defining the Thurston Bennequin invariant, the invariant is independent of the surface bound by the knot - we can write $\text{tb}(K)$. In the general case this is no-longer true.)

- The rotation invariant $\mathrm{rot}(K, [\Sigma], (\mathfrak{s}, h))$ is analogous to the term $c_1(\mathfrak{s}) \cdot [\Sigma]$ in the adjunction inequality. It's defined to be the pairing

$$c_1(L_{\mathfrak{s}}, v) \cdot [\Sigma]$$

  where $L_{\mathfrak{s}}$ is the determinant line bundle of $\mathfrak{s}$. $c_1(L_{\mathfrak{s}}, v)$ is a relative first Chern class, where the boundary condition $v$ is a non-vanishing tangent vector field to $K$. To interpret this as a section of $L_{\mathfrak{s}}$, we use the isomorphism $h$ between $\mathfrak{s}|_Y$ and $\mathfrak{s}_\xi$. (In the 4-ball case, the invariant again doesn't depend on the choice of surface $\Sigma$, and we can write $\mathrm{rot}(K)$.)

## 3. A brief proof outline

- Gluing: Let $(\overline{M}, Y, \xi)$ and $(\overline{M}', Y', \xi')$ be 4-manifolds with contact boundary. Moreover, suppose $\overline{M}' = \overline{M} \cup \overline{Z}$ where $\overline{Z}$ is a special cobordism from $Y$ to $Y'$. Then there's a canonical map

$$j : \mathrm{Spin}^c(\overline{M}, \xi) \to \mathrm{Spin}^c(\overline{M}', \xi'),$$

  and

$$SW_{\overline{M},\xi} \circ j = \pm SW_{\overline{M}',\xi'}.$$

- Consider a surface $\Sigma$ in $\overline{M}$, where $\Sigma$ has boundary a Legendrian knot in $Y = \partial\overline{M}$. We glue a special cobordism $\overline{Z}$ to $\overline{M}$, together with a surface $F$ in $\overline{Z}$.
- The result is a closed surface $\Sigma'$ in a 4-manifold with boundary $\overline{M}'$. We relate the homological invariants of the closed surface $\Sigma'$ to the Thurston Bennequin and rotation invariants of $\Sigma$.
- Since the Seiberg-Witten invariants of $\overline{M}$ are non-zero if and only if the corresponding invariants are non-zero in $\overline{M}'$, the slice Bennequin inequality follows from the adjunction inequality for closed surfaces in 4-manifolds with boundary.

## 4. Gluing

We now describe the gluing result in a little more detail. To begin with, what exactly are we gluing?

**Definition 4.1.** A *special symplectic cobordism* is a symplectic cobordism with some additional geometric and homological constraints. Specifically:

(1) $(\overline{Z}, \omega)$ is a symplectic cobordism if it has boundary $-Y \sqcup Y'$, where $(Y, \xi)$ and $(Y', \xi')$ are contact manifolds, weakly concave and convex respectively. (That is, $\omega$ is strictly positive on $\xi$ and $\xi'$.)
(2) A *special* symplectic cobordism additonally requires that the concave boundary is *strongly concave*. That is, a (collar) neighbourhood of the boundary is given by the symplectisation of the contact structure. (That is, it looks like $Y \times (0, \infty)$, where the symplectic form is given by $\omega = d(t\xi)$.)

(3) A *special* symplectic cobordism also requires a cohomological triviality condition:

$$i^* : H^1(\overline{Z}, Y') \to H^1(Y)$$

must be the zero map. This is essentially saying that any homological aspects of $Y$ are not detected in $\overline{Z}$.

Symplectic cobordisms have a key property that we'll be using: they have canonical Spin$^c$ structures.

- A result by Gompf (which I haven't read) implies that for 4-manifolds with boundary, there's a canonical bijective correspondence between homotopy classes of almost complex structures and spin$^c$ structures.
- Moreover, there are isomorphisms, unique up to homotopy, between $\mathfrak{s}_\omega|_Y$ and $\mathfrak{s}_\xi$, and between $\mathfrak{s}_\omega|_{Y'}$ and $\mathfrak{s}_{\xi'}$.

Now consider the spaces $(\overline{M}, \xi)$ and $(\overline{M} \cup \overline{Z}, \xi')$. Let $(\mathfrak{s}, h)$ be in Spin$^c(\overline{M}, \xi)$. We can glue $\mathfrak{s}$ to $\mathfrak{s}_\omega$ via the isomorphisms

$$\mathfrak{s}|_Y \xrightarrow{h} \mathfrak{s}_\xi \to \mathfrak{s}_\omega|_Y.$$

This gives a new spin$^c$ structure

$$(\mathfrak{s}', h') \in \mathrm{Spin}^c(\overline{M}', \xi').$$

In summary, we have a gluing map

$$j : \mathrm{Spin}^n(\overline{M}, \xi) \to \mathrm{Spin}^c(\overline{M}', \xi').$$

Let's recall the gluing theorem:

**Theorem 4.2.** *Given $(\overline{M}, Y, \xi)$ and a special symplectic cobordism $(\overline{Z}, Y, Y')$, the Seiberg Witten invariants for $\overline{M}' = \overline{M} \cup \overline{Z}$ are given by*

$$SW_{\overline{M},\xi} \circ j = \pm SW_{\overline{M}',\xi'}.$$

*Here $j$ is the canonical map between the respective spin$^c$ structures of $\overline{M}$ and $\overline{M}'$.*

*Proof.* The proof is hard! It's about 50 pages, and I honestly didn't read it. I did however try to extract why we need the additional properties of special symplectic cobordisms, rather than just symplectic cobordisms.

The main idea is that they work at the level of equations and moduli spaces. Most of the work is in establishing a map between the moduli spaces of solutions to the Seiberg Witten equations on $\overline{M}$ and $\overline{M}'$.

(1) The homological condition of special symplectic cobordisms serves only to simplify some algebra, and doesn't have analytic consequences. Specifically, even without the homological condition, a similar consequence to the above theorem holds:

$$\sum_{(\mathfrak{s},h)\in j^{-1}(\mathfrak{s}',h')} SW_{\overline{M},\xi}(\mathfrak{s}, h) = \pm SW_{\overline{M}',\xi'}(\mathfrak{s}', h').$$

(2) The fact that the concave end of the cobordism is given by the symplectisation of the contact boundary is used on page 64 of Mrowka-Rollin. It appears to allow very explicit computations (where they genuinely express the symplectic form in terms of the contact structure) to deduce that the moduli spaces of solutions to the Seiberg-Witten equations are diffeomorphic.

□

## 5. From Adjunction to slice Bennequin

The main (so-far unmentioned) result we'll use to derive the slice Bennequin inequality from the Adjunction inequality is *Weinstein surgery.*

**Theorem 5.1.** *(This is slightly informal), but a specific case of Weinstein surgery is gluing 2-handles along Legendrian knots in the contact boundaries of 4-manifolds. The handles themselves are guaranteed to be special symplectic cobordisms.*

We also recall the version of the adjunction inequality that we'll use:

**Theorem 5.2.** *Let $(\overline{M}, Y, \xi)$ be a 4-manifold with contact boundary, and $SW_{\overline{M},\xi}(\mathfrak{s}, h) \neq 0$. Then for any closed surface of positive genus,*

$$-\chi(\Sigma) \geq |c_1(\mathfrak{s}) \cdot \Sigma|.$$

Finally, we restate the result we'll prove:

**Theorem 5.3.** *Let $(\overline{M}, Y, \xi)$ be a 4-manifold with contact boundary, and suppose $(\mathfrak{s}, h) \in$ $\mathrm{Spin}^c(\overline{M}, \xi)$ satisfies $SW_{\overline{M},\xi}(\mathfrak{s}, h) \neq 0$. Then any $\Sigma$ bounding a Legendrian knot in $Y$ satisfies*

$$-\chi(\Sigma) \geq \mathrm{tb}(K, [\Sigma]) + |\,\mathrm{rot}(K, [\Sigma], (\mathfrak{s}, h))|.$$

We now give a proof outline!

(1) Given the data of $(\overline{M}, Y, \xi)$ and an embedded surface $\Sigma$ with boundary a Legendrian knot, do Weinstein surgery along the knot. This produces a new 4-manifold with boundary $(\overline{M}', Y', \xi')$ by gluing a special symplectic cobordism. Moreover, the core of the handle from Weinstein surgery is a disk being glued to $\Sigma$, which produces a closed surface in $\Sigma'$.
(2) Suppose $(\mathfrak{s}, h) \in \mathrm{Spin}^c(\overline{M}, \xi)$ satisfies $SW_{\overline{M},\xi}(\mathfrak{s}, h) \neq 0$. By the gluing formula, we also know that $SW_{\overline{M}',\xi'}(j(\mathfrak{s}, h)) \neq 0$.
(3) Since $\Sigma'$ was obtained from $\Sigma$ by gluing a disk, we have

$$\chi(\Sigma') = \chi(\Sigma) + 1, \quad [\Sigma']^2 = \mathrm{tb}(K, [\Sigma]) - 1, \quad \langle c_1(j(\mathfrak{s})), [\Sigma'] \rangle = \mathrm{rot}(K, [\Sigma], \mathfrak{s}, h).$$

(4) If $[\Sigma'] = 0$ and $\Sigma'$ has genus at least 1, then the adjunction inequality applies! By applying the above identities, we can deduce the slice Bennequin inequality.
(5) In general, neither of the above facts are true. We can resolve this by replacing $K = \partial\Sigma$ with a new knot, $K_1$. Specifically: let $T$ be a right handed trefoil in the boundary of $B^4$, and $K_1 = K\#T \subset \overline{M}\natural B^4$. The right handed trefoil bounds a

surface of genus 1 in the 4-ball, so we also replace the surface $\Sigma$ with $\Sigma_1$. Notice that

$$g(\Sigma_1) = g(\Sigma) + 1.$$

(6) We also understand how the classical invariants behave under connected sums: the rotation invariant is additive, while the Thurston Bennequin invariant is "additive plus 1". Since the rotation invariant of the right handed trefoil is 0, while the Thurston Bennequin invariant is 1, with each iterated trefoil-gluing operation, we have:

- the genus increases by 1
- the Thurston Bennequin invariant increases by 2
- the rotation invariant is unchanged.

Thus iterating the process enough times ensures that eventually the premise of the adjunction inequality must hold, and so the slice-Bennequin inequality for $K_n$ must hold. Finally, relating each invariant of $K_n$ to $K$ shows that the slice-Bennequin inequality also holds for $K$.