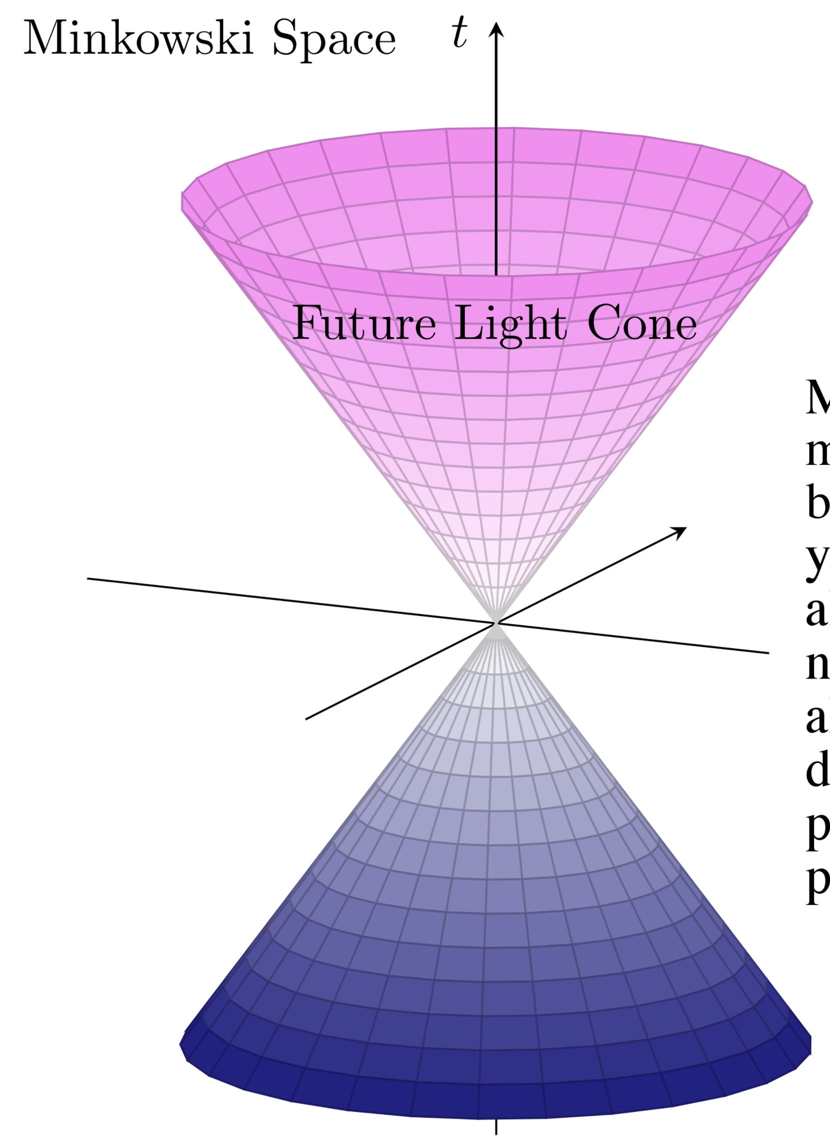
# HOW DO SURFACES IN SPACETIME BEHAVE AT INFINITY?

## BACKGROUND

What is the geometry of the world? We have asked this question for centuries - initially about the geometry of Earth, and now about the geometry of the universe.

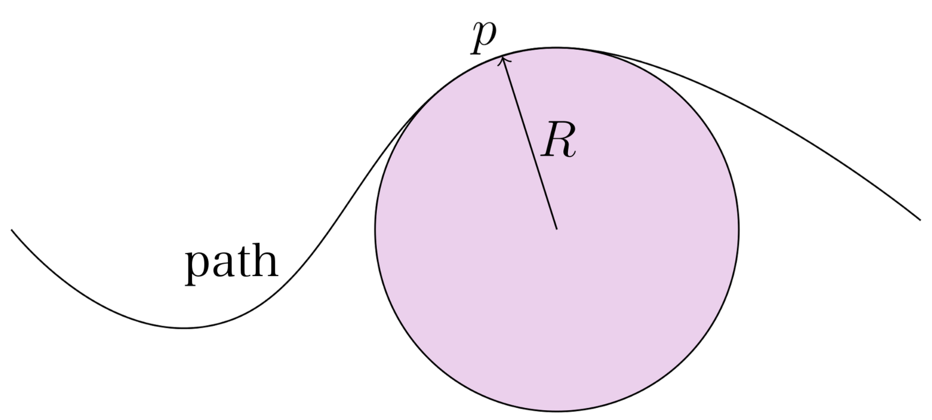
Allen et al. [1] have shown the existence of a useful class of constant mean curvature (CMC) surfaces. We have studied the geometries of the corresponding CMC surfaces in Minkowski Space. In particular, we have investigated the curvature of these surfaces at infinity. This could shed light on the global geometry of the universe [2]. So, what is Minkowski space, and what is constant mean curvature?



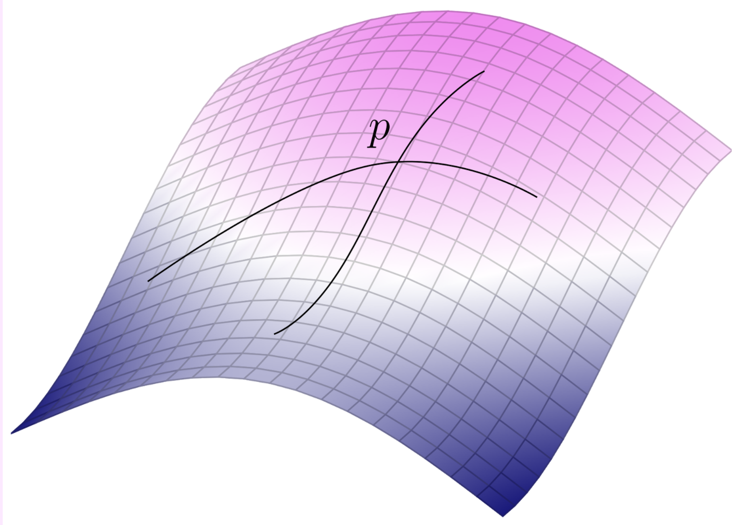

### WHAT IS MINKOWSKI SPACE?

Minkowski space is the simplest model for spacetime, characterised by its unusual notion of distance. If you are at the origin of the graph, all points inside the cone are a negative distance away from you, all points on the cone are zero distance away from you, and all points outside the cone are a positive distance away from you.

### WHAT IS CONSTANT MEAN CURVATURE?




Curvature at $p$ is $\kappa_p = \frac{1}{R}$

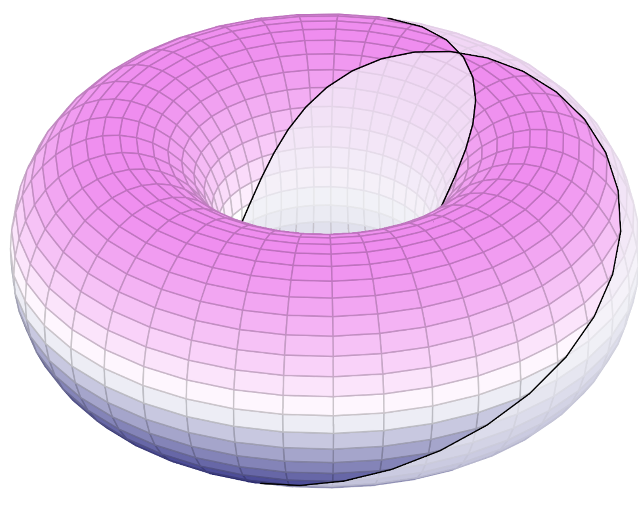



Given a point on a surface one can consider all of the paths on the surface through the point. The average of the curvatures of all of the paths through the point is the *mean curvature* at the point. If the mean curvature is the same everywhere on a surface, we say it has a *constant mean curvature*.

## METHODS

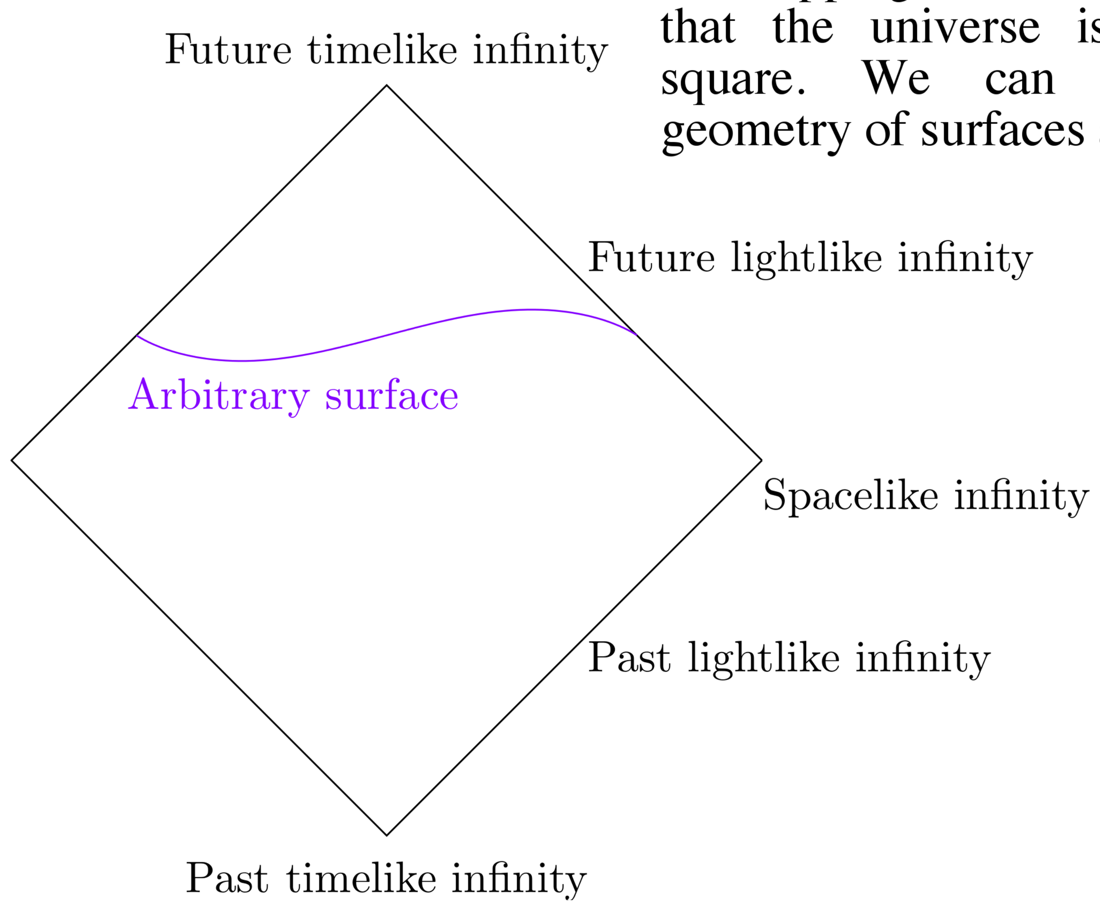
Studying infinity is difficult. It is, by definition, incredibly far away. How do we get around this?

### COMPACTIFYING MINKOWSKI SPACE



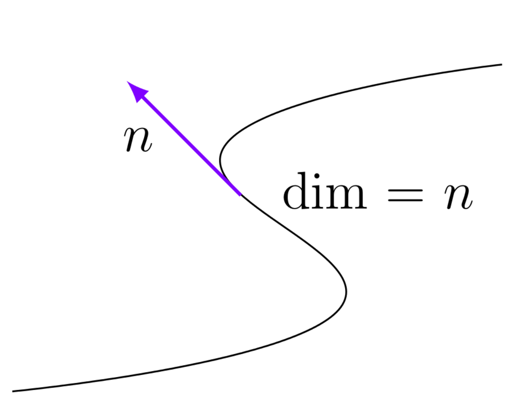
Rather than attempting to travel to infinity, we use a process called conformal compactification to bring infinity to us. In this process, lengths may change, but all angles in spacetime are constant. Specifically, we wrapped Minkowski space around a donut. Minkowski space is the highlighted region, and infinity is the black curve.

Unwrapping this construction, we see that the universe is sitting inside a square. We can now study the geometry of surfaces at infinity.

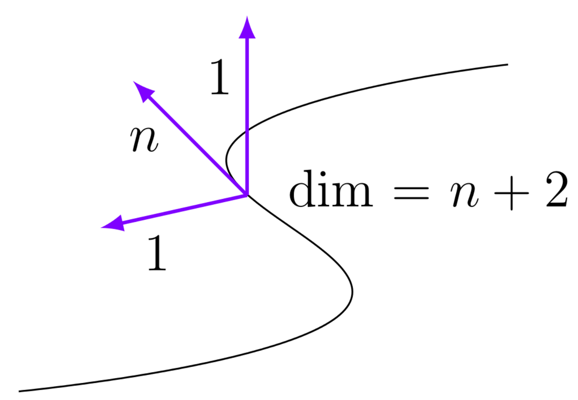



### USING TRACTOR CALCULUS

Compactification alone is not enough. We must keep track of precisely how spacetime is deformed, and this results in a mess of equations. However, we can use tractor calculus [3] to solve this problem. This approach to geometry adds two more dimensions to our space to hide away the complications arising from compactification.

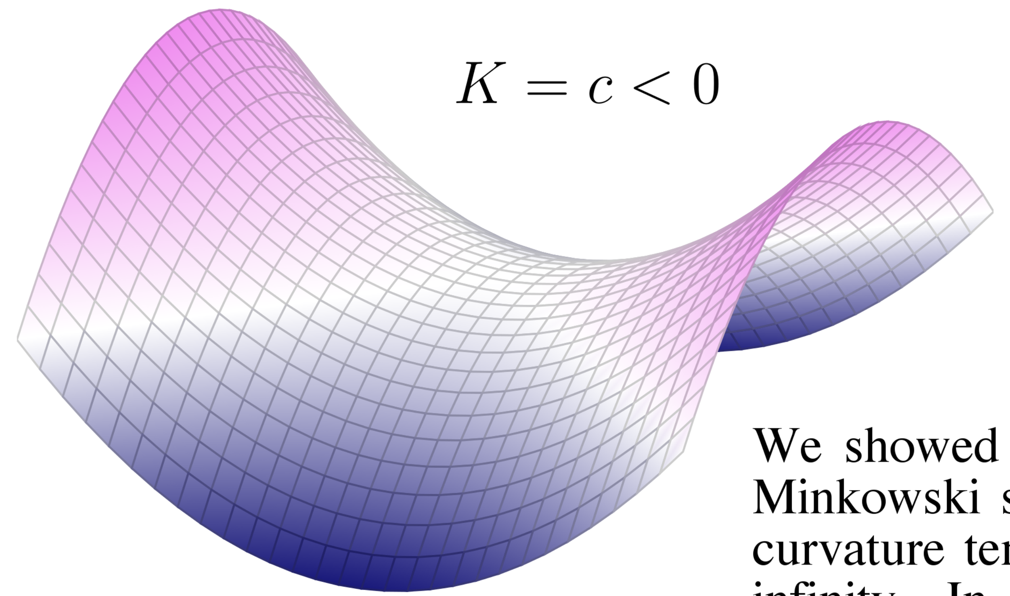



Traditional Approach




Tractor Approach

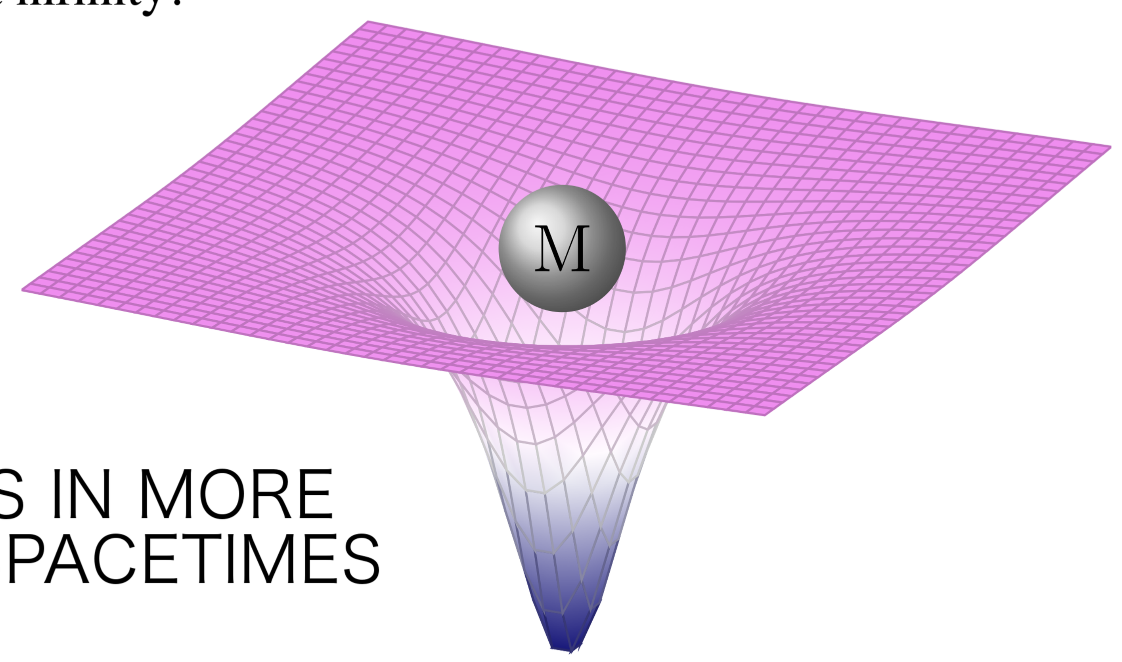
## RESULTS




### CMC SURFACES ARE HYPERBOLIC AT INFINITY

We showed that any CMC surface in Minkowski space with non-zero mean curvature tends to hyperbolic space at infinity. In other words, eventually every point looks like a saddle point. Mathematically this means the sectional curvature, K, tends to a negative constant number. This result shows that it may be possible to understand the geometry of "most" spacetimes at infinity [2].

Our result only holds in Minkowski space, which has a notable property called flatness. Physically, this corresponds to a universe without mass. Since the real universe has mass, the next step is to determine whether or not more general curved spacetimes exhibit similar properties at infinity.




### SURFACES IN MORE GENERAL SPACETIMES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[1] P.T. Allen, J. Isenberg, J.M. Lee and I.S. Allen. *The shear-free condition and constant-mean-curvature hyperboloidal initial data*. 2016 Class. Quantum Grav. 33 115015.

[2] J.M. Lee, private email correspondence.

[3] S. Curry and A. R. Gover. *An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity*. In: ArXiv e-prints (Dec. 2014). arXiv: 1412.7559 [math.DG].