# PARSING “TETRAHEMIHEXAHEDRON”

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Abstract. Join for an interactive afternoon of Euclidean geometry. We'll classify a couple of families of polyhedra (primarily focusing on convex deltahedra). We'll also discuss what words like ”deltahedra” even mean, because there are so many shapes on wikipedia and they all have such absurd names. All of this will happen interactively; we'll all be making paper shapes along the way.

## 1. Platonic solids

10 minutes Introduce the talk, and introduce the first task (cutting paper!). Provide paper templates and scissors to the audience. The talk consists of two themes running in parallel: classifying various families of polyhedra, and inspecting the names (nomenclature) of polyhedra.

### 1.1. **Classification.** There are five platonic solids:

(1) tetrahedron
(2) cube
(3) octahedron
(4) dodecahedron
(5) icosahedron

Draw these on the board! Coloured chalk!! Don't name them yet, just draw them, and leave space to label them underneath for later!

What exactly is a platonic solid? Why are there exactly five?

**Definition 1.1.** A *platonic solid* is a *convex regular polyhedron*. Here *convex* basically means it's orb shaped, and *regular* means that every face is a congruent regular polyhedron, with every vertex meeting the same number of faces.

Definition should be written like a tree!

So why are there five of them? We'll now prove this: it follows a very general and useful proof structure that permeates topology.

(1) Constrain the possibilities (i.e. find an upper bound).
(2) Exhibit/construct possibilities (i.e. find a lower bound).

If you can make the upper and lower bounds agree, then you've found every possible object in your family.

**Constraining possibilities**

(1) $V - E + F = 2$, where $V$ is the number of vertices, $E$ the number of edges, and $F$ the number of faces.
(2) Being a platonic solid means being *regular*, so every vertex meets $d$ edges for some number $d$. On the other hand every edge meets exactly two vertices, so $dV = 2E$.
(3) Similarly, every face meets $n$ edges for some number $n$, and every edge meets two faces, so $nF = 2E$.
(4) We can now write

$$V - E + F = \frac{2E}{d} - \frac{E}{1} + \frac{2E}{n} = 2.$$

(5) Dividing by $2E$ and rearranging, we get the inequality

$$\frac{1}{d} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}.$$

(6) Solving for all $d, n$ at least three for which $1/d + 1/n > 1/2$, there are only five possiblities:

$$(d, n) = (3, 3), (3, 4), (4, 3), (3, 5), (5, 3).$$

**Exhibiting possibilities**

Show audience five straw/string models of platonic solids, thus exhibiting existence for all five $(d, n)$-pairs.

1.2. **Nomenclature.** These five platonic solids are called the *tetrahedron, cube (hexahedron), octahedron, dodecahedron, and icosahedron*, respectively. These names suck! They have the following format:

[greek numerical prefix] + hedron (base/seat).

It's true that for each number 4, 6, 8, 12, and 20, these are the unique convex regular polyhedra with these number of faces. However, without the additional convexity and regularity assumptions, these shapes aren't unique.

5 minutes Exercise: construct a convex dodecahedron with triangular faces.

## 2. Convex deltahedra

5 minutes

**Definition 2.1.** A *deltahedron* is a polyhedron whose faces are equilateral triangles.

**Example.** The regular tetrahedron, octahedron, and icosahedron are all deltahedra. We also just constructed a *dodecadeltahedron*.

2.1. **Classification.** We again wish to constrain possibilities and exhibit possibilities to classify all the convex deltahedra.

**Constraining possibilities**

(1) The *Gauss-Bonnet theorem* in differential geometry says that

$$\int_{\Sigma} \kappa = 4\pi,$$

where $\kappa$ is the curvature of a topological sphere $\Sigma$.

(2) There is a combinatorial analogue: given a polyhedron whose faces are all triangles,
$$\sum_{v \in P} \kappa(v) = 4 \times 3,$$
where $\kappa(v)$ is the "combinatorial curvature" of the polyhedron at the vertex $v$.

(3) The combinatorial curvature is defined by
$$\kappa(v) = 6 - \deg(v),$$
where $\deg(v)$ is the number of edges meeting the vertex $v$. This definition is sensible because the average corner of a triangle is 60°, so on average six triangles meet at a vertex in a flat way. On the other hand, fewer than six triangles meet with positive curvature, and more than six meet with negative curvature.

We'll use this combinatorial Gauss-Bonnet theorem to classify convex deltahedra:

(1) *Convexity* means that every vertex must have positive curvature, so $\kappa(v) \geq 1$.
(2) On the other hand, being a polyhedron means every vertex must meet at least three edges, so $\kappa(v) = 6 - \deg(v) \leq 6 - 3 = 3$.
(3) We solve the equation $\sum \kappa(v) = 12$ with the constraints $1 \leq \kappa(v) \leq 3$. This has exactly 19 solutions!

Unfortunately 19 isn't a good bound—in reality many of these 19 numerical solutions don't correspond to actual polyhedra. However, we do have a finite list to check, and in practice it's not too hard to eliminate the impossible cases one by one. Here's the actual list:

(1) 4; tetrahedron
(2) 6; ?
(3) 8; octahedron
(4) 10; ?
(5) 12; dodecadeltahedron $\sim$ snub disphenoid
(6) 14; ?
(7) 16; ?
(8) 20; icosahedron

10 minutes Task: with neighbours, try to construct the convex deltahedra exhibiting 6, 10, 14, and 16 faces.

2.2. **Nomenclature.** 10 minutes Do we observe any patterns in the shapes? (Discussion)

- Pyramids and bipyramids. The tetrahedron is a *triangular pyramid*. The 6,8, and 10-hedra are *triangular, square,* and *pentagonal bipyramids*. This is because they have the form of two pyramids glued along their bases.
- Augmentation. The 14-hedron is a *tri-augmented* triangular prism.
- Elongation, gyroelongation. The 16-hedron is a square bipyramid pulled apart with a ring of triangles added to *gyroelongate* it. (It's a *gyroelongated square bipyramid.*)
- Rectification (truncation). The octahedron is actually a rectified tetrahedron.
- Snubbification. The 12-hedron is a *snub disphenoid.*

## 3. Non-convex polyhedra

10 minutes We mentioned right at the start that the five *platonic solids* are the unique *convex regular polyhedra*. We also just spent a while thinking about *convex deltahedra*. What about non-convex polyhedra? It turns out that there are several non-convex regular polyhedra:

**Theorem 3.1.** *There are nine regular polyhedra; five convex ones (the platonic solids) and four non-convex.*

we provide some examples and constructions:

- Small stellated dodecahedron. (Stellation is a non-unique operation on polyhedra; the "first" stellation of a dodecahedron is called the small stellated dodecahedron.) Its faces are pentagrams. This is a regular polyhedron, but non-convex.
- Tetrahemihexahedron. Given a polyhedron with an even number of faces (each face with a parallel face on the other side), the "hemification" (hemipolyhedron) is obtained by replacing every pair of parallel faces with a single face of the same shape through the center of the polyhedron. For example, a hemicube (hemihexahedron) is the intersection of three squares, mutually orthogonal. This isn't a polyhedron because not every edge meets two faces. The *tetrahemihexahedron* is a polyhedron obtained from a hemihexahedron by adding four triangles around the hemihexahedron.