Introduction to Topological Quantum Field Theory

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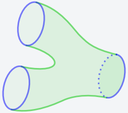

May 29, 2020

Outline for section 1

1. Physics: observe some crucial properties of Feynman's *path integrals*.
2. Category theory: develop a categorical frame work with the desired properties of path integrals.
3. Very low dimensional TQFT: introduce the definition, and study 1 and 2 dimensional TQFTs.
4. Slightly low dimensional TQFT: explore some applications of TQFT.

Path integral definition

Definition

A *path integral* from φ_0 to φ_1 is

$$A(\varphi_0, t_0; \varphi_1, t_1) = \int_{\Phi|_{t_i}=\varphi_i} D[\Phi] e^{iS[\Phi]}.$$

- φ_0 and φ_1 represent *states* in a Hilbert space $\mathcal{H}$.
- A is a *propagator*, gives the likelihood of φ_0 evolving to φ_1.
- The integral is taken over all *field configurations with boundary data φ_i at t_i*.
- $D[\Phi]$ is some measure. S is an *action*.

Path integral pros and cons

- There is a correspondence

$$\text{path integral formalism} \quad \leftrightsquigarrow \quad \text{operator formalism}$$

- Path integrals don't make sense!

Path integral properties i, and ii

i. At time t_0, we obtain a time-constant slice V_0. We expect a corresponding Hilbert space $\mathcal{H}_{V_0} = Z(V_0)$.

ii. A cobordism (M, V_0, V_1) between V_0 and V_1 should correspond to the path integral

$$Z(M)(- \otimes -) = \int_{(\Phi|_{t_0}, \Phi|_{t_1}) = (-,-)} D[\Phi] e^{iS[\Phi]}.$$

More suggestively, we should obtain a *propagator* $Z(M) : \mathcal{H}_{V_0} \otimes (\mathcal{H}_{V_1})^* \to \mathbb{C}$. Equivalently,

$$Z(M) : \mathcal{H}_{V_0} \to \mathcal{H}_{V_1}.$$

Path integral properties iii and iv

iii. If V_0 and V_0' are disjoint, then

$$\mathcal{H}_{V_0 \sqcup V_0'} = \mathcal{H}_{V_0} \otimes \mathcal{H}_{V_0'}.$$

iv. A cylindrical cobordism $M = V_0 \times [t_0, t_1]$ corresponds to the propagator $Z(M) : \mathcal{H}_{V_0} \to \mathcal{H}_{V_0}$.

For φ_0 normalised we expect

$$A(\varphi_0, \varphi_0) \sim 1.$$

Therefore $Z(M) : \mathcal{H}_{V_0} \to \mathcal{H}_{V_0}$ should be the identity.

Path integral property v: sewing law

v. (Sewing law.) For $t_0 < t' < t_1$ we expect

$$\int_{\Phi|_{t_i}=\varphi_i} D[\varphi]e^{iS[\varphi]} = \int_{\varphi' \text{ at } t'} D[\varphi']\left(\int_{\Phi|_{t_i}=\varphi_i, \Phi|_{t'}=\varphi'} D[\Phi]e^{iS[\Phi]}\right).$$

This corresponds to

$$Z(M) = Z(M_1)Z(M_0)$$

where M is a cobordism from V_0 to V_1, with $M = M_0 M_1$.

Outline for section 2

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Braided monoidal categories

Definition

A *braided monoidal category* is a category $\mathcal{C}$ equipped with a "tensor product". More precisely, $\mathcal{C}$ is equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which

- has a unit: $1 \in \mathcal{C}$ such that $x \otimes 1 \cong 1 \otimes x \cong x$,
- is associative: $x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$,
- and has a *braiding* $B_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$.

There are additional "coherence conditions" for the natural isomorphisms (requiring that certain diagrams commute).

Symmetric monoidal categories

Definition

A braided monoidal category is called a *symmetric monoidal category* if the braiding is involutive:

$$B_{x,y} \circ B_{y,x} = \mathrm{id}_{x \otimes y} \, .$$

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Example

$\mathbf{Vect}_k$ is a symmetric monoidal category, with the product given by the usual tensor product $\otimes_k$.

Example

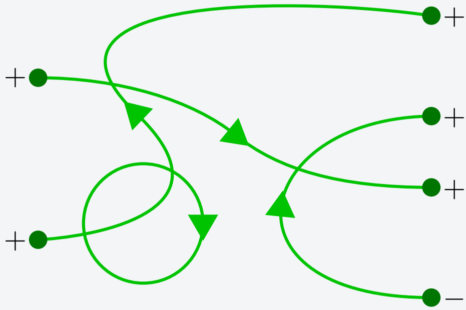
The category $n\mathbf{Cob}$ of oriented n-dimensional cobordisms is a symmetric monoidal category. The product of two closed $(n-1)$-manifolds is given by their *disjoint union*.

A closer look at 1**Cob**

Objects of 1**Cob** are oriented 0-manifolds, i.e. finite disjoint unions of signed points:

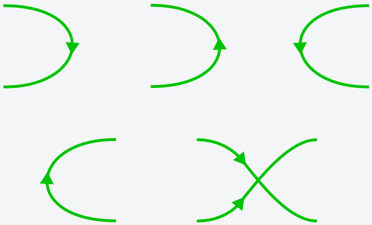
$$\varnothing, \quad +, \quad + \sqcup - \sqcup -, \quad +^n \sqcup -^m.$$

The morphisms are (oriented) 1-manifolds with these points as boundaries. For example,

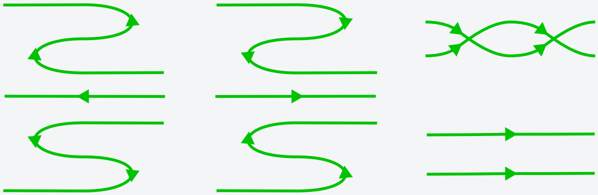


A closer look at 1**Cob**

Generators of 1**Cob**:



Relations in 1**Cob**:



Symmetric monoidal functor definition

Definition

A *symmetric monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories which preserves the product and the braiding.

More precisely, the following diagram commutes:

$$\begin{array}{ccc} F(x \otimes y) & \longrightarrow & F(y \otimes x) \\ \downarrow & & \downarrow \\ F(x) \otimes F(y) & \longrightarrow & F(y) \otimes F(x) \end{array}$$

In addition, F must respect the unit and associativity.

Symmetric monoidal functors from $1\mathbf{Cob}$ to $\mathbf{Vect}_k$

To determine $Z : 1\mathbf{Cob} \to \mathbf{Vect}_k$, the following data suffices:

- $Z(+) = V \in \mathbf{Vect}_k$.
- $Z(-) = W \in \mathbf{Vect}_k$.
- $Z(\varphi)$ for each generator φ.

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- $Z(-) = W \in \mathbf{Vect}_k$.
- $Z(\varphi)$ for each generator φ.

1. Given $Z(+)$, we necessarily have $Z(-) = Z(+)^*$.
2. Moreover, we necessarily have

$Z\Big($  $\Big) : V \otimes V^* \to k, \quad v \otimes \varphi \mapsto \varphi(v)$

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$Z\Big($  $\Big) : k \to V^* \otimes V, \quad \lambda \mapsto \lambda \sum e_i^* \otimes e_i$

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Symmetric monoidal functors from 1**Cob** to $\mathbf{Vect}_k$

3. Since Z is *symmetric*:

$Z\Big($  $\Big) : V \otimes V \to V \otimes V, \quad (v, w) \mapsto (w, v).$

Result:

Every symmetric monoidal functor $Z : 1\mathbf{Cob} \to \mathbf{Vect}_k$ is completely determined by $Z(+)$.

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Result:

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$Z\Big($  $\Big) = Z\Big($  $\Big) \circ Z\Big($  $\Big)$

$$= \left(\varphi \otimes v \mapsto \varphi(v)\right) \circ \left(\lambda \mapsto \lambda \sum e_i \otimes e_i^*\right)$$
$$= \lambda \mapsto \lambda \sum 1$$
$$= \lambda \mapsto (\dim V)\lambda.$$

Outline for section 3

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Definition of a TQFT

Definition

An n-dimensional *topological quantum field theory* is a symmetric monoidal functor

$$Z : n\mathbf{Cob} \to \mathbf{Vect}_k,$$

for some fixed $n \in \mathbb{N}$ and field k.

Theorem

Topological quantum field theories $1\mathbf{Cob} \to \mathbf{Vect}_k$ *are in bijective correspondence with finite dimensional vector spaces over* k. *The correspondence is given by*

$$Z \longmapsto Z(+).$$

Why is the definition good?

i. The functor Z sends each time slice to a space of states; i.e. a vector space.

ii. Z sends a cobordism (M, V_0, V_1) to a linear map $Z(M) : \mathcal{H}_{V_0} \to \mathcal{H}_{V_1}$. (This is the *propagator*.)

iii. Since Z is a symmetric monoidal functor, it indeed sends $Z(V \sqcup V') = Z(V) \otimes Z(V')$.

iv. By functoriality, $Z(M) = \text{id}$ whenever M is a cylinder (trivial cobordism).

v. By functoriality, $Z(M_0 M_1) = Z(M_1) \circ Z(M_0)$, verifying the *sewing law*.

Classification of 2 dimensional TQFTs

Theorem

There is an equivalence of groupoids

$$\{\textit{TQFTs } 2\mathbf{Cob} \to \mathbf{Vect}_k\} \longleftrightarrow \mathbf{comFrob}_k$$

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Definition

A *Frobenius algebra* is an algebra A over a field equipped with a non-degenerate bilinear form

$$\sigma : A \times A \to k, \quad \sigma(ab, c) = \sigma(a, bc).$$

- $\text{Mat}_{n\times n}$ equipped with $\sigma(A, B) = \text{tr}(AB)$.
- $k[G]$ equipped with $\sigma(a, b) =$ coefficient of e in ab.

Frobenius algebras: categorical edition

Definition

A *Frobenius algebra over k* is a vector space A with morphisms

$$\mu : A \otimes A \to A,\ \eta : k \to A; \quad \delta : A \to A \otimes A,\ \varepsilon : A \to k,$$

such that (A, μ, η) is a monoid, (A, δ, ε) is a comonoid, and

$$\delta \circ \mu = (\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta).$$

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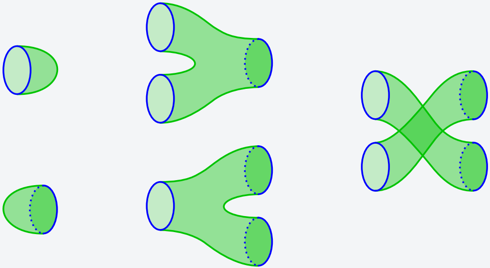
Definition

A *morphism of Frobenius algebras* is a k-linear map preserving both the monoid and comonoid structures.

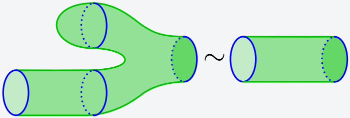
$$\mathbf{comFrob}_k \subset \mathbf{Frob}_k \subset \mathbf{Vect}_k$$

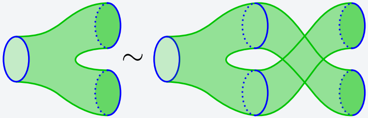
The structure of 2**Cob**

Generators:

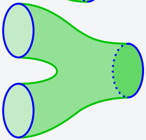
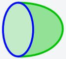
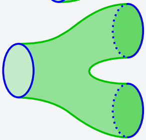


Some relations:

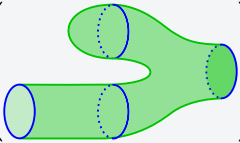


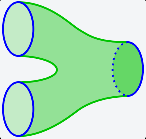
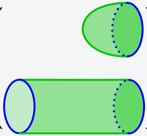


TQFTs $\mathbf{2Cob} \to \mathbf{Vect}_k$

M	$Z(M)$	Interpretation
	$\eta : k \to A$	unit
	$\mu : A \otimes A \to A$	multiplication
	$\varepsilon : A \to k$	counit
	$\delta : A \to A \otimes A$	comultiplication

Is it really a Frobenius algebra?

$\mathrm{id}_A = Z\Big($  $\Big) = Z\Big($  $\Big)$

$= Z\Big($  $\Big) \circ Z\Big($  $\Big)$

$= \mu \circ (\eta \otimes \mathrm{id}_A).$

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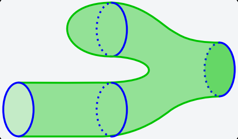
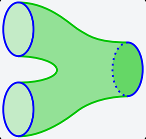
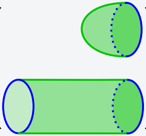
$\mathrm{id}_A = Z\Big($  $\Big) = Z\Big($  $\Big)$

$= Z\Big($  $\Big) \circ Z\Big($  $\Big)$

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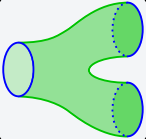
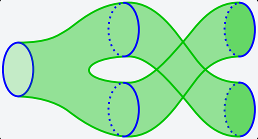
$\delta = Z\Big($  $\Big) = Z\Big($  $\Big) = \beta \circ \delta.$

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Functorial quantum field theory

TQFTs $n\mathbf{Cob} \to \mathbf{Vect}_k$ are a starting point for other "functorial QFTs".

Additional structure on M	Corresponding FQFT
Conformal	Conformal field theory
pseudo-Riemannian	Relativistic QFT
Submanifolds	Defect TQFT
Spin	Spin TQFT
Framing	Framed TQFT

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- 3d TQFT: Chern-Simons theory
- 4d TQFT: Topological Yang-Mills theory

Chern-Simons theory

- Schwarz-type TQFT.
- Action:

$$S[A] = \frac{k}{4\pi} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

  - M is a 3-manifold, with a principal G-bundle $P \to M$. (G is called the *gauge group*.)
  - A is a *connection 1-form*; $A \in \Omega^1(M, \mathfrak{g})$.

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For G abelian, Chern-Simons theories have been formalised as functorial TQFTs (Freed, Hopkins, Lurie, Teleman).

Chern-Simons theory to knot theory

Path integral for $L \subset M$:

$$\int_{\Omega^1(M,\mathfrak{g})} e^{iS[A]} \prod \chi_{L_i}(A)\, dA.$$

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- $G = U(2)$, $M = \mathbb{S}^3 \rightsquigarrow$ Jones polynomial of L.
- $G = U(n)$, $M = \mathbb{S}^3 \rightsquigarrow$ HOMFLY polynomial of L.
- $G = SO(n)$, $M = \mathbb{S}^3 \rightsquigarrow$ Kauffman polynomial of L.

"A TQFT is a QFT that computes topological invariants"