THE UNCERTAINTY PRINCIPLE

SHINTARO FUSHIDA-HARDY

1. Heisenberg uncertainty principle

Suppose $p : \mathbb{R} \to \mathbb{R}$ is a probability density function for a random variable X. The *moments* of X are given by

$$m_n := \mathbb{E}(X^n) = \int_{\mathbb{R}} x^n p(x)\mathrm{d}x. \tag{1}$$

(Assuming they exist), the first moment m_n is the *mean* of X, while $m_2 - m_1^2$ is the *variance* of X. The *Heisenberg uncertainty principle* in quantum mechanics states that

$$\mathrm{Var}(x)\mathrm{Var}(p) \geq \frac{\hbar^2}{4} \tag{2}$$

where x and p denote position and momentum of a particle, respectively. If these both have mean zero, the above equation is

$$\Big(\int_{\mathbb{R}} t^2 p_x(t)\mathrm{d}t\Big)\Big(\int_{\mathbb{R}} s^2 p_p(s)\mathrm{d}x\Big) \geq \frac{\hbar^2}{4}, \tag{3}$$

where p_x and p_p are probability density for position and momentum. But from quantum mechanics, the probability density for position is precisely the wave function of the particle under consideration. Moreover, p_p is the Fourier transform of p_x with additional constants. In this document we prove a more general result from which the Heisenberg uncertainty principle follows.

2. Definitions and Theorems

The *Schwartz space* on $\mathbb{R}$ are is the collection of *rapidly decreasing* complex valued smooth functions on $\mathbb{R}$. Formally,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \|f\|_{\alpha,\beta} < \infty\}, \tag{4}$$

where the collection of norms is defined for all $\alpha, \beta \in \mathbb{N}$ by

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha f^{(\beta)}(x)|. \tag{5}$$

The Schwartz space is the natural space for Fourier analysis, since the *Fourier transform* is a homeomorphism on this space. It also has the delicious property of being contained in $L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$. The Fourier transform is defined by

$$\hat{f}(\zeta) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\zeta}\mathrm{d}x. \tag{6}$$

On $\mathcal{S}(\mathbb{R})$, the inverse is simply

$$f(x) = \int_{\mathbb{R}} \hat{f}(\zeta) e^{2\pi i x \zeta} \mathrm{d}\zeta. \tag{7}$$

Moreover, the *Parseval identity* holds for all $f, g \in \mathcal{S}(\mathbb{R})$:

$$\int_{\mathbb{R}} f(x)\overline{\hat{g}(x)}\mathrm{d}x = \int_{\mathbb{R}} \hat{f}(x)\overline{g(x)}\mathrm{d}x. \tag{8}$$

The *Plancherel identity* immediately follows:

$$\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\hat{f}(x)|^2 \mathrm{d}x. \tag{9}$$

Lemma 2.1. Let $f \in \mathcal{S}(\mathbb{R})$. Then $\hat{f'}(\zeta) = 2\pi i \zeta \hat{f}(\zeta)$.

Proof. This follows from integration by parts:

$$2\pi i \zeta \hat{f}(\zeta) = 2\pi i \zeta \int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta} \mathrm{d}x = -\Big[f(x)e^{-2\pi i x\zeta}\Big]_{-\infty}^{\infty} + \int_{\mathbb{R}} f'(x) e^{-2\pi i x \zeta} \mathrm{d}x.$$

The first term on the right hand side vanishes since f is a Schwartz function. □

3. Fourier uncertainty principle

Recalling the form of the uncertainty principle in equation 2, we prove the following Fourier uncertainty principle:

Theorem 3.1. *Let $f \in \mathcal{S}(R)$. Suppose f is normalised, that is, $\|f\|_2 = 1$. Then*

$$\Big(\int_{\mathbb{R}} x^2 |f(x)|^2 \mathrm{d}x\Big)\Big(\int_{\mathbb{R}} \zeta^2 |\hat{f}(\zeta)|^2 \mathrm{d}\zeta\Big) \geq \frac{1}{16\pi^2}. \tag{10}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}} x f(x)\overline{f'(x)}\mathrm{d}x &= \Big[x f(x)\overline{f(x)}\Big]_{-\infty}^{\infty} - \int_{\mathbb{R}} (x f(x))'\overline{f(x)}\mathrm{d}x \\ &= 0 - \int_R f(x)\overline{f(x)}\mathrm{d}x - \int_R x f'(x)\overline{f(x)}\mathrm{d}x.\end{aligned}$$

The first term on the right side vanishes by virtue of f being a Schwartz function. Rearranging the above gives

$$\begin{aligned}1 = \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x &= -\int_{\mathbb{R}} x(f'(x)\overline{f(x)} + f(x)\overline{f'(x)})\mathrm{d}x \\ &= -2\int_{\mathbb{R}} x \mathrm{Real}[f(x)f'(x)]\mathrm{d}x.\end{aligned}$$

By the Cauchy-Schwartz inequality,

$$1 \leq \Big(2\int_{\mathbb{R}} |x\mathrm{Real}[f(x)f'(x)]|\mathrm{d}x\Big)^2 \leq 4\Big(\int_{\mathbb{R}} |x f(x)|^2 \mathrm{d}x\Big)\Big(\int_{\mathbb{R}} |f'(x)|^2 \mathrm{d}x\Big).$$

Moreover, by lemma 2.1 and the Plancherel identity,

$$\int_{\mathbb{R}} |f'(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\hat{f}'(\zeta)|^2 \mathrm{d}\zeta = \int_{\mathbb{R}} |2\pi i\zeta \hat{f}(\zeta)|^2 \mathrm{d}\zeta = 4\pi^2 \int_{\mathbb{R}} |\zeta \hat{f}(\zeta)|^2 \mathrm{d}\zeta. \tag{11}$$

Combining this with the previous inequality,

$$1 \leq 16\pi^2 \Big(\int_{\mathbb{R}} |xf(x)|^2 \mathrm{d}x \Big) \Big(\int_{\mathbb{R}} |\zeta \hat{f}(\zeta)|^2 \mathrm{d}\zeta \Big), \tag{12}$$

so

$$\Big(\int_{\mathbb{R}} x^2 |f(x)|^2 \mathrm{d}x \Big) \Big(\int_{\mathbb{R}} \zeta^2 |\hat{f}(\zeta)|^2 \mathrm{d}\zeta \Big) \geq \frac{1}{16\pi^2} \tag{13}$$

as required. □

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1142, New Zealand

Email address: `sfus128@aucklanduni.ac.nz`